Dark matter from exact general relativistic thin disks in higher dimensions

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The possibility to construct a galactic disk embedded in a multidimensional space-time is investigated. Particularly we are interested in a disk that lives in a universe endowed with Universal Extra Dimensions. The simplest example is a six dimensional space-time disk constructed by solving the vacuum Einstein equations for an extension of the Weyl's metric. In particular, we study a disk constructed from Schwarzschild and Chazy-Curzon solutions with a simple extension for the extra dimensions. Two integral constants of motion from projection of extradimensional particle velocities are the free parameters of the model. We prevent the *ad hoc* adjustment of such parameters with observed rotation curves, preferring to investigate values where the disk becomes stable. The stability is achieved when the disk is Newtonian-like (where such parameters are null) or for a tiny range of values that astonishingly makes the circular geodesics fit with great precision the rotation curves of many spiral galaxies. The stability calculation is done using both the Rayleigh criterion and a perturbative approach. We compare such results to well succeeded astrophysical dark matter profiles and demonstrate that our predictions give the same gravitational lensing as does a dynamically successful dark halo model. Finally, we consider the possibility that our model could constrain a Kaluza-Klein dark matter particle to be tested at Large Hadron Collider (LHC).

PACS numbers: 04.20.Jb, 04.40.-b, 04.50.+h, 11.25.-w, 95.35.+d

arXiv:astro-ph/0703466v1 18 Mar 2007

I. INTRODUCTION

The possibility that a dark matter (DM) particle could be detected at the Large Hadron Collider (LHC) in next years is believed to be one of the most promising discoveries to be done on experimental particle physics at the present century. It is expected, in particular, that such particle could be a kind of extradimensional dark matter (or a Kaluza-Klein particle). Recently, many important works are being focused on this area, where it is taking place a serious attempt to describe theoretically and phenomenologically the main characteristics of a such particle [1]. The most standard proposed model is the Universal Extradimensional scenario (UED), a braneworld model where all particles move in the whole extradimensional space [2]. However, actually there are few studies about the behavior of those particles in an astrophysical environment as a galactic disk. Rotation curves of galaxies and gravitational lensing of galactic clusters are the main observational tools to understand the presence of DM in universe. Here, as a first step, we investigate a general relativistic galactic disk and calculate the gravitational lensing regime for a galactic cluster living in a universe endowed with UED.

The existence of DM comes from strong observational evidence, primarily from dynamical and lensing effects: galactic disks, cluster of galaxies, and a smoothly distributed cosmological background present undeniable results. In galactic disks, where Newtonian gravitational theory would have been awaited to be an excellent explanation, accelerations of stars and gas, as estimated from Doppler velocities are much larger than those due to the Newtonian field generated by the visible matter in those systems (the plateau anomaly in rotation curves of galaxies) [3]. Rotation curves are the major tool for determining the distribution of mass in spiral galaxies, and are also important to study kinematics and to infer the evolutionary histories in galactic systems. Historically they are the most basic and classic manner to bring on the presence of dark matter in galaxies (for a complete review about rotation curves, see for instance [4]). On the other hand, is verified that cluster of galaxies are composed of three main components: $\sim 5\%$ in mass is the optically luminous baryonic matter in hundred of bright galaxies; ~ 10–15% is in the form of a bright X-ray inter-cluster gas; and the remaining ~ 80–85% is some sort of non-baryonic "missing mass". The first evidence of such "dark matter" in cluster of galaxies was provided by Zwicky in 1937 [5] who verified, by applying the virial theorem to the Coma cluster, that most of the matter in this cluster was dark. Another form to estimate clus-

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ter masses is by using gravitational lensing techniques, for both the strong and weak regimes [6]. When interpreted within $4D$ general relativity (GR), this lensing is atypically large unless one assumes the presence of DM in quantities similar to those demanded to explain the accelerations of stars and gas. Such techniques plus the temperature fluctuations in the Cosmic Background Radiation have been regarded as confirming the DM existence.

Current chiefly astrophysical models used to achieve cosmological results for the evolution of cluster of galaxies are the cold dark matter (CDM) approaches [7]. Here the "missing matter" problem is solved by using a dark halo of exotic postulated particles surrounding galaxies. One potential problem for the ΛCDM model, as for most other versions of CDM models, is that it predicts galaxy halos that have a steeper density profile in the inner parts than may be observed and produces too many small satellites galaxies in contradiction to what is observed [8]. The non-baryonic candidates for particles in such models range e.g. from primordial black holes [9] to elementary particles relics left over from the early universe (e.g. WIMPS [10]). Observational and experimental difficulties to identify such particles raise the possibility that the acceleration discrepancy as well the gravitational lensing anomaly may reflect departures from Newtonian/GR gravity on galactic or cluster scales: it means the theoretical appealing about a formal modification of Newtonian/GR dynamics (e.g. MOND [11] and TeVeS [12]). However, recent results obtained using the merging cluster 1E0657-558, where a combination of a long Chandra X-ray observation with accurate strong and weak gravitational lensing maps, are allowing some authors to claim about a first direct proof of the existence of dark matter, dismissing a modified gravity paradigm [13].

Another way to explain dark matter is assuming that the universe is endowed with extra dimensions and we are living in a $3+1$ membrane (or brane) [14]. In a such conception, the "missing mass" problem comes from the gravitational interaction among galaxies located at distinct parts of the brane or comes directly from possible fields named extradimensional modes (or Kaluza-Klein modes). Concerning extra dimensions, in present days there is a considerable activity that is motivated by possible extra dimensions arising within a TeV scale for quantum gravity, and in spite of this, there are basic questions concerning the nature of gravity in higher dimensions that remain unanswered. Classically an initial attempt to include extra dimensions in a physical theory was made by Kaluza and Klein in 1920s [15], where they suggested an unification of electromagnetism and gravity into a geometrical formulation, involving one extra dimension that, when compactified, makes a unique formalism that unravels the Lorentz and the U(1) gauge symmetry of electromagnetism simultaneously. Braneworld scenarios [14] and superstring theories [16] are the main current models that advocate extra dimensions to explain some of the problems present in the standard model of particles, and open subjects in cosmology as dark energy and dark matter. However, these subjects have the tendency to explain the nature using a bottom-up design, where first the foundations are specified by the axiomatic existence of fundamental strings whose vibrating modes form all the physical particles.

Here, following the observational evidence of DM in galaxies and clusters, we investigate the possibility that DM comes from extradimensional modes, or (KK modes), although initially we do not know if those modes are distributed isotropically on the galaxy disk or are described by a spheroidal halo (as preached by CDM cosmologies). Assuming first the "isotropic on the disk" scenario, it is important to investigate solutions of Einstein field equations in axially symmetric configurations in D dimensions to construct a disk model composed of Kaluza-Klein particles. Exact solutions of Einstein field equations in axially symmetric configurations are an important tool to understand the dynamical properties of real systems which can be described approximately by a thin disk. In this sense, axially symmetric solutions have great astrophysical interest, because they can be used to model galaxies and accretion disks. A reasonably accurate general relativistic model of a spiral galaxy would require an exact solution of Einstein field equations that describes a superposition of a Kerr black hole with a stationary disk, a central bulge with or without an external exotic halo and the presence of magnetic fields. A long range of disk solutions was derived with or without radial pressure. Solutions for static disks without radial pressure were first studied by Bonnor and Sackfield [17], and Morgan and Morgan [18], and with radial pressure by Morgan and Morgan [19]. Static thin disks without radial stress generate a Weyl spacetime. Then they are described by two different metric functions [20]. The stability of these models can be explained by either assuming the existence of hoop stresses or that the particles on the disk plane move under the action of their own gravitational field in such a way that as many particles move clockwise as counterclockwise. This last interpretation is frequently made since it can be invoked to mimic true rotational effects. A large class of static thin disks solutions were obtained by Letelier and Oliveira [21] using the inverse scattering method. Disks with radial tension have been considered in [22], and disk models with electric fields [23] and magnetic fields [24], and both magnetic and electric fields [25]. Solutions for self-similar static disks were analyzed by Lynden-Bell and Pineault

[26], and Lemos [27]. Another relevant approaches are the superposition of static disks with black holes or disk solutions with halos [28], and see other important astrophysical solutions to mimic AGNs in [29]. Recently Vogt and Letelier refined general relativist models of galaxies in a considerable way [30]. Important discussions about the role of general relativistic disks to explain rotation curves of galaxies can be found, e.g., in [31] and essential counter-arguments in [32]. On axisymmetric geometries in D dimensions, generalized Weyl solutions in D dimensions were presented by Emparan and Reall [33] and important objects were classified by them, as black rings and Kaluza-Klein bubbles. About disk rotation curves in higher dimensions see for instance [34]; this Ref. contains the method to derive the Platonic Model, a model where KK fields do not need to live in a external halo to reproduce the rotation curves of galactic disks and in galaxy clusters the modes will follow the general spheroidal geometry by the superposition of individual modes on galaxies. Also matter densities on galaxies are following the KK distribution. For other important works about solutions of Einstein equations in D dimensions see [35].

Here we construct a UED disk by solving the vacuum Einstein equations for an extension of the Weyl's metric. The main purpose of the work is to construct a top-down model, where we are interested in what is coming to pass in large scales assuming a multidimensional scenario. The simplest example in $6D$ is presented and the disk is constructed from Schwarzschild and Chazy-Curzon solutions. These solutions are employed to generate a disk by introducing discontinuities in the first derivatives of the metric functions (image method). Two integral constants of motion from projection of extradimensional particle velocities are the free parameters of the model. We prevent the *ad hoc* adjustment of such parameters with observed rotation curves, preferring to investigate values where the disk becomes stable. The stability is achieved when the disk is Newtonian-like (where such parameters are null) or for a tiny range of values that astonishingly makes the circular geodesics fit with great precision the rotation curves of many spiral galaxies. We compare such results to well succeeded astrophysical dark matter profiles and demonstrate that our predictions give the same gravitational lensing as does a dynamically successful dark halo model. We conclude that the modes are not "isotropic on the disk" but the KK particle distribution follows the matter distribution. This is barely sufficient to achieve our results. Another interpretation is that what really is happening is that matter is following the KK distribution, endorsing the Platonic Model approach.

The present work is organized as follows: in Section II we present the principal aspects of static thin disk solutions in $4D$ and the expression for the energy-momentum tensor of the disk. In Section III, we derive general Weyl's solutions. To derive Weyl's solutions to higher dimensions is to find all solutions of the vacuum Einstein equations that admit $D-2$ orthogonal commuting Killing vector fields. We show, following [33], that in D dimensions, Einstein equations in a Weyl geometry is reduced to the Laplace's equations, what permits to use the image method to construct a disk living in a D dimensional universe. In Section IV we exemplify by constructing a simple multidimensional disk model using six dimensional axially symmetric Einstein equations in vacuum, with a "Schwarzschild solution" for the four dimensional part; the extra dimensional part is solved with a Chazy-Curzon solution. These solutions are employed to generate a disk by introducing discontinuities in the first derivatives of the metric functions (image method). In Section V we find a equation for the circular geodesic orbits (rotation curves) and study the stability of such orbits in Section VI both by the Rayleigh criterion and by a perturbative method. Results and graphics for stable rotation curves and comparison to observed rotation curves of spiral galaxies are showed in Section VII. In Section VIII we derive the gravitational lensing regime of the model to a spherically symmetric cluster of galaxies. Discussions about such results is done in the Concluding Remarks (Section IX) where we comment about the astrophysical Navarro-Frenk-White CDM profiles and point out on astrophysical and cosmological issues (apart from the question of cosmological dark matter which is left open). And although the main purpose of the present work is to construct a top-down model and not to shed light on a fundamental theory we also illustrate considering the possibility that our model could constrain a Kaluza-Klein dark matter particle to be tested at Large Hadron Collider (LHC) in next years, where we write considerations about the compactification of a $6D$ space-time following the example presented. In what follows, we do $c = G = 1$.

II. THIN DISKS IN $4D$

Much endeavor has been strongly attached to advance in techniques for finding general relativistic exact solutions in four dimensions [36, 37]. One of the earliest accomplishments in this direction was attained by Weyl [20], who found the general static axisymmetric solution of the vacuum Einstein equations:

$$\mathrm{d}s^2 = -e^{2\Phi}\mathrm{d}t^2 + e^{-2\Phi}\left(e^{2\gamma}(\mathrm{d}r^2+\mathrm{d}z^2)+r^2\mathrm{d}\varphi^2\right), \quad (2.1)$$

where $\Phi(r,z)$ is an arbitrary axisymmetric solution of Laplace's equation in a three-dimensional *flat* space with line element

$$\mathrm{d}\sigma^2 = \mathrm{d}r^2 + r^2\mathrm{d}\varphi^2 + \mathrm{d}z^2, \tag{2.2}$$

and γ satisfies

$$\gamma_{,r} = r[\Phi_{,r}^2 - \Phi_{,z}^2], \tag{2.3}$$

$$\gamma_{,z} = 2r\Phi_{,r}\Phi_{,z}, \tag{2.4}$$

where $(\ )_{,a} = \partial/\partial x^a$. The solution of these equations is given by a line integral. Since Φ is harmonic, it can be regarded as a Newtonian potential produced by certain (axisymmetric) sources. Since in this coordinate the spherically symmetric solutions of the Einstein equations correspond to a bar of density $1/2$, one needs to be careful in the use of Newtonian images [21, 45].

As an illustration, the metric (2.1) for a static axially symmetric $4D$ space-time can be written in quasicylindrical coordinates (r, φ, z) in the form

$$\mathrm{d}s^2 = -e^{-\phi}\mathrm{d}t^2 + \chi^2 e^{\phi}\mathrm{d}\varphi^2 + f(dr^2 + dz^2), \tag{2.5}$$

where χ, ϕ, and f are functions of r and z only. In the vacuum, the Einstein equations are equivalent to

$$\chi_{,rr} + \chi_{,zz} = 0, \tag{2.6}$$

$$(\chi\phi_{,r})_{,r} + (\chi\phi_{,z})_{,z} = 0. \tag{2.7}$$

Let $\zeta = r + iz$. It is possible to consider χ as the real part of an analytical function $W(\zeta) = \chi(r,z) + iZ(r,z)$. Noting that $\overline{W}(\zeta) = W(\overline{\zeta})$, one can write $\mathrm{d}W\mathrm{d}\overline{W} = \frac{\partial W}{\partial \zeta}\frac{\partial \overline{W}}{\partial \overline{\zeta}}\mathrm{d}\zeta\mathrm{d}\overline{\zeta} = |W'(\zeta)|^2\mathrm{d}\zeta\mathrm{d}\overline{\zeta}$. Or even, $\mathrm{d}W\mathrm{d}\overline{W} = \mathrm{d}\chi^2 + \mathrm{d}Z^2 = |W'(\zeta)|^2(\mathrm{d}r^2 + \mathrm{d}z^2)$. Thus without lossing generality one can choose $\chi = r$. In such a way we can write $f(r,z)$ as a functional of ϕ

$$\begin{aligned} \ln f[\phi] \ &= \ \frac{1}{2}\int r\{[\phi_{,r}^2 - \phi_{,z}^2]\mathrm{d}r \\ &+ \ [2\phi_{,r}\phi_{,z}]\mathrm{d}z\}. \end{aligned} \tag{2.8}$$

In order to obtain a solution of (2.6)–(2.8) representing a thin disk located at $z = 0$, we assume that the metric functions χ, ϕ, and f are continuous across the disk, but have discontinuous first derivatives in the direction normal to the disk. Thin disk solutions in Weyl coordinates are functions of the class C^0. The reflectional symmetry of (2.6) - (2.8) with respect to the plane $z = 0$ allows us to assume that χ, ϕ, and f are even functions of z. Hence, $\chi_{,z}$, $\phi_{,z}$ and $f_{,z}$ are odd functions of z. We shall require that they not vanish on the surfaces $z = 0^{\pm}$. Such impositions can induce e.g. a discontinuity in the space-time by reflecting the solution through the plane. This represents the construction of the disk using the well known "displace, cut and reflect" method that was used by Kuzmin [38] in Newtonian gravity and later in GR by many authors [22]-[30]. The material content of the disk will be described by functions that are distributions with support on the disk. The method can be divided in the following steps that are illustrated in Fig. 1. First, in a space wherein we have a compact source of gravitational field, we choose a surface (in our case, the plane $z = 0$) that divides the space in two pieces: one with no singularities or sources and the other with the sources. Then we disregard the part of the space with singularities and use the surface to make an inversion of the non-singular part of the space. This results in a space with a singularity that is a delta function with support on $z = 0$.

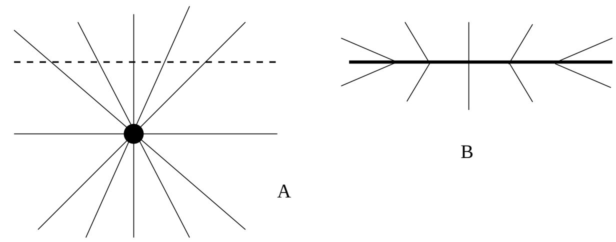


FIG. 1: Illustration of the "displace, cut and reflect" method for the generation of disks. In A the spacetime with a singularity is displaced and cut by a plane (dotted line), in B the part with singularities is disregarded and the upper part is reflected on the plane.

This procedure is mathematically equivalent to make the transformation $z \to |z| + a$, with a constant. In the Einstein tensor we have first and second derivatives of z. Since $\partial_z|z| = 2\vartheta(z) - 1$ and $\partial_{zz}|z| = 2\delta(z)$, where $\vartheta(z)$ and $\delta(z)$ are, respectively, the Heaviside function and the Dirac distribution. Therefore the Einstein field equations will separate in two different pieces [41]: one valid for $z \neq 0$ (the usual Einstein equations), and other involving distributions with an associated energy-momentum tensor. Due to the discontinuous behavior of the derivatives of the metric tensor across the disk, the Riemann curvature tensor contains Dirac delta functions. The energy-momentum tensor can be obtained by the distributional approach due to Papapetrou and Hamouni [39], Lichnerowicz [40], and Taub [41]. It can be written as $T^a{}_b = [T^a{}_b]\ \delta(z)$, where δ is the Dirac function with support on the disk and $[T^a{}_b]$ is the distributional energy-momentum tensor, which yield the volume energy density and the principal stresses. A different approach is given by Israel [42] where one makes use of the extrinsic curvature of the surface to represent the matter. Here, $a, b = 0, ..., 3$.

A. Density and pressures on the disk

The disk at $z = 0$ divides the space-time into two halves. The normal to the disk can be described by the co-vector $n_a = \partial z/\partial x^a = (0, 0, 0, 1)$. Above the disk near $z = 0$, we can expand the metric as

$$g_{ab} = g^0_{ab} + z\frac{\partial g^+_{ab}}{\partial z}|_{z=0} + z^2\frac{\partial^2 g^+_{ab}}{\partial z^2}|_{z=0} + ..., \tag{2.9}$$

and below $z = 0$,

$$g_{ab} = g^0_{ab} + z\frac{\partial g^-_{ab}}{\partial z}|_{z=0} + z^2\frac{\partial^2 g^-_{ab}}{\partial z^2}|_{z=0} + ... \tag{2.10}$$

The quantity g^0_{ab} means the value of g_{ab} at $z = 0$. The discontinuities in the first derivatives of the metric tensor can be cast as

$$b_{ab} \;=\; g_{ab,z}\big|_{z=0^+} \;-\; g_{ab,z}\big|_{z=0^-}. \tag{2.11}$$

The symmetry of the problem gives $\phi^+_{,z}|_{z=0} = -\phi^-_{,z}|_{z=0}$, and $f^+_{,z}|_{z=0} = -f^-_{,z}|_{z=0}$. Denoting $\phi_{,z}|_{z=0} = \phi^+_{,z}|_{z=0}$, and the same for the other functions, we calculate the following discontinuities:

$$b_{tt} = 2\mathrm{e}^{-\phi}\phi_{,z}|_{z=0}, \tag{2.12}$$

$$b_{rr} = 2f_{,z}|_{z=0}, \tag{2.13}$$

$$b_{zz} = b_{rr}, \tag{2.14}$$

$$b_{\varphi\varphi} = 2r^2\mathrm{e}^{\phi}\phi_{,z}|_{z=0}. \tag{2.15}$$

It is possible to find the other discontinuities terms by contracting indices in b^{ab}:

$$b^{tt} = -2\mathrm{e}^{\phi}\phi_{,z}|_{z=0} \quad ; \quad b^t{}_t = 2\phi_{,z}, \tag{2.16}$$

$$b^{rr} = -\frac{2}{f^2}f_{,z}|_{z=0} \quad ; \quad b^r{}_r = -\frac{2}{f}f_{,z}, \tag{2.17}$$

$$b^{zz} = -\frac{2}{f^2}f_{,z}|_{z=0} \quad ; \quad b^z{}_z = -\frac{2}{f}f_{,z}, \tag{2.18}$$

$$b^{\varphi\varphi} = -\frac{2}{r^2}\mathrm{e}^{-\phi}\phi_{,z}|_{z=0} \quad ; \quad b^\varphi{}_\varphi = -2\phi_{,z}. \tag{2.19}$$

From (2.11), one can work out (the discontinuities of) the Christoffel symbols through the disk given by

$$[\Gamma^a{}_{bc}] = \frac{1}{2}(b^a{}_c\delta^z{}_b + b^a{}_b\delta^z{}_c - g^{az}b_{bc}) \tag{2.20}$$

where $[\Gamma^a{}_{bc}] \equiv \Gamma^{+a}{}_{bc} - \Gamma^{-a}{}_{bc}$ at $z = 0$. From the Riemann tensor defined by

$$\begin{aligned} R_{abcd} = \;& \frac{1}{2}(g_{ad,bc} - g_{bd,ac} + g_{bc,ad} - g_{ac,bd}) \\ & + g_{rs}\Gamma^r{}_{ad}\Gamma^s{}_{bc} - g_{rs}\Gamma^r{}_{ac}\Gamma^s{}_{bd}, \end{aligned} \tag{2.21}$$

we can compute the Riemann distributional tensor,

$$[R^a{}_{bcd}] = \frac{1}{2}(b^a{}_d\delta^z{}_b\delta^z{}_c - b^a{}_c\delta^z{}_b\delta^z{}_d + g^{az}b_{bd}\delta^z{}_c). \tag{2.22}$$

Defining the Ricci distributional tensor as $[R_{ab}] = [R^c{}_{acb}]$ and the Ricci distributional scalar $[R] = [R^a{}_a]$, we can identify the distributional energy-momentum tensor on the disk through Einstein equations as

$$[R^a{}_b] - \frac{1}{2}\delta^a{}_b[R] = 8\pi[T^a{}_b]. \tag{2.23}$$

Then the distributional energy-momentum tensor is given by

$$\begin{aligned} [T^a{}_b] \;&= \frac{1}{16\pi}\{b^{az}\delta^z{}_b - b^{zz}\delta^a_b + g^{az}b^z{}_b - g^{zz}b^a{}_b \\ & + b^c{}_c(g^{zz}\delta^c{}_b - g^{az}\delta^z{}_b)\}. \end{aligned} \tag{2.24}$$

This energy-momentum tensor describes the matter content (fluid) of a thin disk located on $z = 0$. The components of a such tensor are calculated as

$$[T^t{}_t] = \frac{1}{16\pi}\{-b^{zz} + g^{zz}(b^r{}_r + b^z{}_z)\}, \tag{2.25}$$

$$[T^r{}_r] = \frac{1}{16\pi}\{-b^{zz} + g^{zz}(b^t{}_t + b^\varphi{}_\varphi)\}, \tag{2.26}$$

$$[T^z{}_z] = 0, \tag{2.27}$$

$$[T^\varphi{}_\varphi] = \frac{1}{16\pi}\{-b^{zz} + g^{zz}(b^t{}_t + b^r{}_r)\}. \tag{2.28}$$

Defining the vielbein

$$\begin{aligned} {e_{(t)}}^a &= \left(\frac{1}{\sqrt{-g_{tt}}}, 0, 0, 0\right), {e_{(r)}}^a = \left(0, \frac{1}{\sqrt{g_{rr}}}, 0, 0\right), \\ {e_{(\varphi)}}^a &= \left(0, 0, \frac{1}{\sqrt{g_{\varphi\varphi}}}, 0\right), {e_{(z)}}^a = \left(0, 0, 0, \frac{1}{\sqrt{g_{zz}}}\right), \end{aligned}$$

one can write down the energy-momentum tensor (2.24) as

$$\begin{aligned} [T^a{}_b] = \; & - \; \epsilon {e_{(t)}}^a {e_{(t)}}^b + p_r {e_{(r)}}^a {e_{(r)}}^b \\ & + \; p_\varphi {e_{(\varphi)}}^a {e_{(\varphi)}}^b + p_z {e_{(z)}}^a {e_{(z)}}^b, \end{aligned} \tag{2.29}$$

yielding the volume densities, i.e., the energy density and pressures as

$$\epsilon = -[T^t{}_t] = -\frac{f_{,z}}{8\pi f^2}|_{z=0} \tag{2.30}$$

$$p_\varphi = [T^\varphi{}_\varphi] = -\frac{\phi_{,z}}{8\pi f}|_{z=0} \tag{2.31}$$

$$p_r = [T^r{}_r] = 0, \tag{2.32}$$

$$p_z = [T^z{}_z] = 0. \tag{2.33}$$

In Fig. 2 we show the density ϵ and pressure p_φ profiles of the $4D$ constructed thin disk.

B. Solutions

In Eq. (2.5), the function ϕ is intrinsically related to the Newtonian potential U by $\phi = 2U$. A important property of Weyl's metric is the fact that Eq. (2.7) to be a Laplace's equation in cylindrical coordinates, and exploiting the characteristics of its linearity it is possible to employ the superposition of solutions. Some of the asymptotically plane solutions are, e.g., Chazy-Curzon, the finite rod, and the Schwarzschild ones.

In the Chazy-Curzon case, the solution for a particle of mass m in the position $z = z_0$ is given by [43, 44]

$$\phi = \frac{2m}{R}, \qquad \ln f = \frac{m^2 r^2}{R^4}, \tag{2.34}$$

where $R = \sqrt{r^2 + (z - z_0)^2}$.

The Schwarzschild solution corresponds to taking the source for ϕ to be a thin rod on the z-axis with $1/2$ linear mass density.

C. Rotation curves

In first approximation one can consider that the particles of such above fluid move along geodesics. In particular, we can consider particles moving along circular geodesics whose tangential velocities give us the rotation curves.

From Eq. (2.5) we have the first integral of motion,

$$-\,\mathrm{e}^{-\phi}\dot{t}^2 + f(\dot{r}^2 + \dot{z}^2) + r^2\mathrm{e}^{\phi}\dot{\varphi}^2 = 1, \tag{2.35}$$

where $\dot{x}^a = \mathrm{d}x^a/\mathrm{d}s$. Assuming $\dot{r} = 0$ and $\dot{z} = 0$ (particles with no radial motion and confined on $z = 0$), Eq.(2.35) reads

$$-\,\mathrm{e}^{-\phi}\dot{t}^2 + r^2\mathrm{e}^{\phi}\dot{\varphi}^2 = 1. \tag{2.36}$$

The geodesic equations on the disk reduce to

$$(\mathrm{e}^{-\phi})_{,r}\dot{t}^2 - (r^2\mathrm{e}^{\phi})_{,r}\dot{\varphi}^2 = 0. \tag{2.37}$$

Eqs. (2.36) and (2.37) form a system of equations for $\dot{\varphi}^2$ and $\dot{t}^2$. From these equations we find the rotation curves V_C,

$$V_C = \sqrt{-\frac{g_{\varphi\varphi}}{g_{tt}}}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \sqrt{-\frac{g_{\varphi\varphi}}{g_{tt}}\frac{\dot{\varphi}^2}{\dot{t}^2}}, \tag{2.38}$$

reduce to

$$V_C = \sqrt{\phi_{,r}/(2/r + \phi_{,r})}. \tag{2.39}$$

In Fig. 3 we show some rotation curves for the constructed $4D$ disk, where we are varying the cut disk parameter a. Here we are applying the thin rod solution with linear mass density of $1/2$ (Schwarzschild solution).

III. WEYL SOLUTIONS FOR ANY D

The first action to generalize Weyl's geometry to spacetimes endowed with higher dimensions is to consider a suitable coordinate chart for the D-dimensional line element. Here we follow the method presented by [33], a simple generalization of what is done in four dimensions [37]. It will be alleged that the metric is Riemannian or Lorentzian. Let $\xi_{(i)}$ represent the Killing vector fields, $1 \leq i \leq D-2$. It is admissible to elect coordinates (x^i, y^1, y^2) such that $\xi_{(i)} = \partial/\partial x^i$, since these commute, with the metric coefficients depending only on y^1 and y^2.

Now we must show that one can select coordinates y^1 and y^2 to span two-dimensional surfaces orthogonal to all of the $\xi_{(i)}$. In order to perform this, one has to make evident that the two-dimensional subspaces of the tangent space orthogonal to all of the vectors $\xi_{(i)}$ are integrable. Sufficient conditions for integrability are afforded by the following theorem:

Theorem (Emparan-Reall). Let $\xi_{(i)}$, $1 \leq i \leq D-2$ be commuting Killing vector fields such that for each i, (a) $\xi_{(1)}^{[\mu_1}\xi_{(2)}^{\mu_2}\dots\xi_{(D-2)}^{\mu_{D-2}}\nabla^\nu\xi_{(i)}^{\rho]}$ vanishes at at least one point of the spacetime (not necessarily the same point for every i), and (b) $\xi_{(i)}^\nu R_\nu^{[\rho}\xi_{(1)}^{\mu_1}\xi_{(2)}^{\mu_2}\dots\xi_{(D-2)}^{\mu_{D-2}]} = 0$. Then the two planes orthogonal to the $\xi_{(i)}$ are integrable.

Condition (b) is trivially satisfied if only vacuum solutions of the Einstein equations are contemplated. About condition (a), in four dimensions it is usually supposed that one of the Killing vector fields is an angular coordinate that corresponds to rotations about an axis of symmetry, and must therefore vanish on this axis. This attests that condition (a) is fulfilled. For the case of higher dimensions, if the conditions of this theorem are met then the coordinates y^1 and y^2 can be chosen in one of the orthogonal surfaces and then extended along the

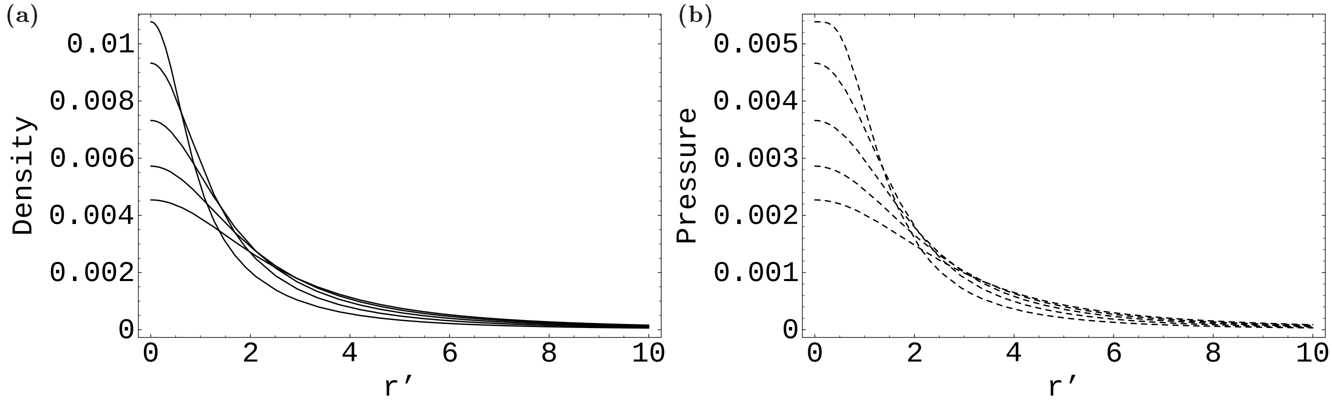


FIG. 2: **(a)** $4D$ disk surface density per unit mass profiles for disk cut parameters $a = 1, 1.5, 2, 2.5, 3$ (from top to bottom). We take $r' = r/m$. **(b)** The same as **(a)** but for disk pressures.

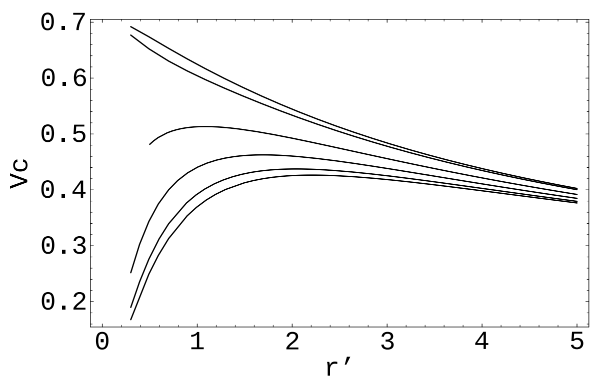


FIG. 3: $4D$ disk rotation curves with cut parameters from $a = 0.3$ to $a = 2$ (from top to bottom). These curves have a Newtonian-like profile. We take $r' = r/m$.

integral curves of the Killing vector fields. In this coordinate system, the vectors $\partial/\partial y^i$ are orthogonal to $\partial/\partial x^j$. If it is further assumed that the Killing vector fields are orthogonal to each other then the metric must take the form

$$\mathrm{d}s^2 = \sum_{i=1}^{D-2} \epsilon_i e^{2\Phi_i} (\mathrm{d}x^i)^2 + g_{ab} \mathrm{d}y^a \mathrm{d}y^b, \tag{3.40}$$

where a and b take the values $1, 2$, the metric coefficients are independent of x^i, and $\epsilon_i = \pm 1$ according to whether $\xi_{(i)}$ is spacelike or timelike. Locally it is always possible to find coordinates such that

$$g_{ab} \mathrm{d}y^a \mathrm{d}y^b = e^{2C} \mathrm{d}Z \mathrm{d}\bar{Z}, \tag{3.41}$$

where Z and $\bar{Z}$ are complex conjugate coordinates if the transverse space is spacelike. The function C is independent of x^i.

Now in order to calculate the curvature tensors, introduce a vielbein for the metric (3.40), considering the coordinates of Eq. (3.41):

$$e^i = e^{\Phi_i} \mathrm{d}x^i, \qquad e^Z = e^C \mathrm{d}Z, \quad e^{\bar{Z}} = e^C \mathrm{d}\bar{Z}. \tag{3.42}$$

The summation convention is not being used for the indices $i, j \ldots$. The tangent space metric $\eta_{\alpha\beta}$ is given by $\eta_{ii} = \epsilon_i$, $\eta_{Z\bar{Z}} = \eta_{\bar{Z}Z} = 1/2$, where the other components vanish. Now the connection 1-forms are defined as

$$\mathrm{d}e^\alpha = -\omega^\alpha{}_\beta \wedge e^\beta. \tag{3.43}$$

From (3.42) and (3.43) we get

$$\omega_{iZ} = e^{-C} \partial_Z \Phi_{i,Z} e_i, \quad \omega_{i\bar{Z}} = e^{-C} \partial_{\bar{Z}} \Phi_i e_i, \quad \omega_{ij} = 0, \tag{3.44}$$

$$\omega_{Z\bar{Z}} = -\frac{1}{2} e^{-C} \partial_Z C e^Z + \frac{1}{2} e^{-C} \partial_{\bar{Z}} C e^{\bar{Z}}. \tag{3.45}$$

The curvature 2-forms are defined by

$$\Theta_{\alpha\beta} = \mathrm{d}\omega_{\alpha\beta} + \omega_\alpha{}^\gamma \wedge \omega_{\gamma\beta}. \tag{3.46}$$

The non-vanishing curvature 2-forms for the connections (3.44) and (3.45) are

$$\Theta_{ij} = -2e^{-2C} \left(\partial_Z \Phi_i \partial_{\bar{Z}} \Phi_j + \partial_Z \Phi_j \partial_{\bar{Z}} \Phi_i \right) e_i \wedge e_j, \tag{3.47}$$

$$\Theta_{iZ} = -e^{-2C}\left[\partial_Z^2\Phi_i + (\partial_Z\Phi_i)^2 - 2\partial_Z C\partial_Z\Phi_i\right] e_i \wedge e^Z - e^{-2C}\left[\partial_Z\partial_{\bar{Z}}\Phi_i + \partial_Z\Phi_i\partial_{\bar{Z}}\Phi_i\right] e_i \wedge e^{\bar{Z}}, \tag{3.48}$$

$$\Theta_{i\bar{Z}} = -e^{-2C}\left[\partial_Z\partial_{\bar{Z}}\Phi_i + \partial_Z\Phi_i\partial_{\bar{Z}}\Phi_i\right] e_i \wedge e^Z - e^{-2C}\left[\partial_{\bar{Z}}^2\Phi_i + (\partial_{\bar{Z}}\Phi_i)^2 - 2\partial_{\bar{Z}} C\partial_{\bar{Z}}\Phi_i\right] e_i \wedge e^{\bar{Z}}, \tag{3.49}$$

$$\Theta_{Z\bar{Z}} = e^{-2C}\partial_Z\partial_{\bar{Z}}Ce^Z \wedge e^{\bar{Z}}. \tag{3.50}$$

The tangent space components of the Riemann tensor are derived from these expressions and

$$\Theta_{\alpha\beta} = \frac{1}{2}R_{\alpha\beta\gamma\delta}e^\gamma \wedge e^\delta. \tag{3.51}$$

We find

$$R_{ijkl} = -2e^{-2C}\left(\partial_Z\Phi_i\partial_{\bar{Z}}\Phi_j + \partial_Z\Phi_j\partial_{\bar{Z}}\Phi_i\right)\left(\eta_{ik}\eta_{jl} - \eta_{il}\eta_{jk}\right), \tag{3.52}$$

$$R_{iZjZ} = -e^{-2C}\left[\partial_Z^2\Phi_i + (\partial_Z\Phi_i)^2 - 2\partial_Z C\partial_Z\Phi_i\right]\eta_{ij}, \tag{3.53}$$

$$R_{i\bar{Z}j\bar{Z}} = -e^{-2C}\left[\partial_{\bar{Z}}^2\Phi_i + (\partial_{\bar{Z}}\Phi_i)^2 - 2\partial_{\bar{Z}} C\partial_{\bar{Z}}\Phi_i\right]\eta_{ij}, \tag{3.54}$$

$$R_{iZj\bar{Z}} = -e^{-2C}\left(\partial_Z\partial_{\bar{Z}}\Phi_i + \partial_Z\Phi_i\partial_{\bar{Z}}\Phi_i\right)\eta_{ij}, \tag{3.55}$$

$$R_{Z\bar{Z}Z\bar{Z}} = e^{-2C}\partial_Z\partial_{\bar{Z}}C, \tag{3.56}$$

with any other non-vanishing components related to these by the symmetries of the Riemann tensor. Afterwards the non-vanishing components of the Ricci tensor are calculated,

$$R_{ij} = -2e^{-2C}\left[2\partial_Z\partial_{\bar{Z}}\Phi_i + \partial_Z\Phi_i\sum_k\partial_{\bar{Z}}\Phi_k + \partial_{\bar{Z}}\Phi_i\sum_k\partial_Z\Phi_k\right]\eta_{ij}, \tag{3.57}$$

$$R_{ZZ} = -e^{-2C}\sum_i\left(\partial_Z^2\Phi_i + (\partial_Z\Phi_i)^2 - 2\partial_Z C\partial_Z\Phi_i\right), \tag{3.58}$$

$$R_{\bar{Z}\bar{Z}} = -e^{-2C}\sum_i\left(\partial_{\bar{Z}}^2\Phi_i + (\partial_{\bar{Z}}\Phi_i)^2 - 2\partial_{\bar{Z}} C\partial_{\bar{Z}}\Phi_i\right), \tag{3.59}$$

$$R_{Z\bar{Z}} = -e^{-2C}\left[2\partial_Z\partial_{\bar{Z}}C + \sum_i\partial_Z\partial_{\bar{Z}}\Phi_i + \sum_i\partial_Z\Phi_i\partial_{\bar{Z}}\Phi_i\right]. \tag{3.60}$$

Now, assuming as before that $\mu, \nu = i, Z, \bar{Z}$, then the vacuum Einstein equations read $R_{\mu\nu} = 0$. The ij component gives

$$\partial_Z\left[\exp\left(\sum_j\Phi_j\right)\partial_{\bar{Z}}\Phi_i\right] + \partial_{\bar{Z}}\left[\exp\left(\sum_j\Phi_j\right)\partial_Z\Phi_i\right] = 0. \tag{3.61}$$

Summing this equation over i yields

$$\partial_Z\partial_{\bar{Z}}\exp\left(\sum_j\Phi_j\right) = 0. \tag{3.62}$$

This last has as general solution

$$\sum_j\Phi_j = \log\left(w(Z) + \tilde{w}(\bar{Z})\right), \tag{3.63}$$

where $\tilde{w} = \bar{w}$ if Z and $\bar{Z}$ are complex conjugate. Substituting equation (3.63) into equation (3.61) yields

$$2(w+\tilde{w})\partial_Z\partial_{\bar{Z}}\Phi_i + \partial_Z w\partial_{\bar{Z}}\Phi_i + \partial_{\bar{Z}}\tilde{w}\partial_Z\Phi_i = 0. \quad (3.64)$$

If w is non-constant then $R_{ZZ} = 0$ can be rearranged to give

$$\partial_Z C = \frac{\sum_i \partial_Z^2\Phi_i}{\sum_i \partial_Z\Phi_i} + \frac{1}{2}\sum_i \partial_Z\Phi_i - \frac{\sum_{i<j}\partial_Z\Phi_i\partial_Z\Phi_j}{2\sum_i\partial_Z\Phi_i}. \quad (3.65)$$

An akin equation is obtained from $R_{\bar{Z}\bar{Z}} = 0$:

$$\partial_{\bar{Z}} C = \frac{\sum_i \partial_{\bar{Z}}^2\Phi_i}{\sum_i \partial_{\bar{Z}}\Phi_i} + \frac{1}{2}\sum_i \partial_{\bar{Z}}\Phi_i - \frac{\sum_{i<j}\partial_{\bar{Z}}\Phi_i\partial_{\bar{Z}}\Phi_j}{2\sum_i\partial_{\bar{Z}}\Phi_i}. \quad (3.66)$$

Integrating the first two terms of these equations and applying equation (3.63) we get

$$C = \frac{1}{2}\log\left(\partial_Z w\partial_{\bar{Z}}\tilde{w}\right) + \Xi, \quad (3.67)$$

where

$$\partial_Z\Xi = -\frac{w+\tilde{w}}{\partial_Z w}\sum_{i<j}\partial_Z\Phi_i\partial_Z\Phi_j, \quad (3.68)$$

$$\partial_{\bar{Z}}\Xi = -\frac{w+\tilde{w}}{\partial_{\bar{Z}}\tilde{w}}\sum_{i<j}\partial_{\bar{Z}}\Phi_i\partial_{\bar{Z}}\Phi_j. \quad (3.69)$$

The integrability condition for Ξ is

$$\partial_Z\partial_{\bar{Z}}\Xi = \partial_{\bar{Z}}\partial_Z\Xi. \quad (3.70)$$

It is straightforward to check that this equation is indeed satisfied if equations (3.63) and (3.64) hold. These equations also confirm that the remaining Einstein equation $R_{Z\bar{Z}} = 0$ is satisfied.

Since w and $\tilde{w}$ have been assumed non-constant, one can perform a coordinate transformation from Z and $\bar{Z}$ to $w(Z)$ and $\tilde{w}(\bar{Z})$ in the similar way as in the $4D$ case mentioned at the beginning of Sec. II. In $4D$, they are referred as "Weyl's canonical coordinates" [36]. This yields

$$\mathrm{d}s^2 = \sum_i \epsilon_i e^{2\Phi_i}(\mathrm{d}x^i)^2 + e^{2\,\Xi}\mathrm{d}w\mathrm{d}\tilde{w}. \quad (3.71)$$

This coordinate transformation is conformal. Eqs. (3.64), (3.68) and (3.69) are conformally invariant so the transformation just replaces ∂_Z by $\partial \equiv \partial_w$ and $\partial_{\bar{Z}}$ by $\bar{\partial} \equiv \partial_{\tilde{w}}$. Then the solution is determined by the following equations

$$\sum_i \Phi_i = \log(w+\tilde{w}), \quad (3.72)$$

$$2(w+\tilde{w})\partial\bar{\partial}\Phi_i + \partial\Phi_i + \bar{\partial}\Phi_i = 0, \quad (3.73)$$

$$\partial\Xi = -(w+\tilde{w})\sum_{i<j}\partial\Phi_i\partial\Phi_j, \quad (3.74)$$

$$\bar{\partial}\Xi = -(w+\tilde{w})\sum_{i<j}\bar{\partial}\Phi_i\bar{\partial}\Phi_j. \quad (3.75)$$

If Z and $\bar{Z}$ are complex conjugate coordinates then, as mentioned above, one must take $\tilde{w} = \bar{w}$. Introduce real coordinates (r, z) by $w = r + iz$, so the canonical form of the metric is

$$\mathrm{d}s^2 = \sum_i \epsilon_i e^{2\Phi_i}(\mathrm{d}x^i)^2 + e^{2\,\Xi}(\mathrm{d}r^2 + \mathrm{d}z^2). \quad (3.76)$$

Eq. (3.73) then takes the form

$$\frac{\partial^2\Phi_i}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi_i}{\partial r} + \frac{\partial^2\Phi_i}{\partial z^2} = 0, \quad (3.77)$$

which is again just Laplace's equation in three-dimensional flat space with metric (2.2). The function Φ_i is independent of the coordinate θ, i.e., it is axisymmetric. The solution is therefore specified by $D-3$ independent axisymmetric solutions of Laplace's equation in three-dimensional flat space.

IV. AN EXAMPLE OF $6D$ DISK

The fact that Φ_i, for D dimensions, has as solution axisymmetric solutions of Laplace's equation leads to the possibility to construct a disk with higher dimensional components. To obtain a solution for such a metric which represents a thin disk located on $y^a = z = 0$, we assume the functions of the metric Φ_i and g_{ab} are continuous along the disk, in particular on the surface $z = 0$, but with discontinuous first derivatives on that surface. Following the same approach developed in Sec. II, we can introduce these discontinuities by doing the replacement $z \to |z| + a$, where a is a constant. The metric discontinuities are derived from expansions above the surface $z = 0$,

$$g_{AB} = g^0_{AB} + z\frac{\partial g^+_{AB}}{\partial z}|_{z=0} + z^2\frac{\partial^2 g^+_{AB}}{\partial z^2}|_{z=0} + ..., \quad (4.78)$$

and below $z = 0$,

$$g_{AB} = g^0_{AB} + z\frac{\partial g^-_{AB}}{\partial z}|_{z=0} + z^2\frac{\partial^2 g^-_{AB}}{\partial z^2}|_{z=0} + ... \quad (4.79)$$

Such discontinuities indicate that the Riemann curvature tensor contains Dirac delta functions that makes possible to calculate the energy-momentum tensor by a distributional approach, as before. This tensor can be written as $T^A_B = [T^A_B]\delta(z)$, where $\delta(z)$ is the Dirac Delta function with support on the disk and $[T^A_B]$ is the distributional energy-momentum tensor. Writing the discontinuities of first derivatives of the metric as $b_{AB} = (g^{+}_{AB,z} - g^{-}_{AB,z})|_{z=0}$, the distributional energy-momentum tensor reads

$$[T^A_B] = \frac{1}{16\pi}\{b^{Az}\delta^z_B - b^{zz}\delta^A_B + g^{Az}b^z_B - g^{zz}b^A_B + b^C_C(g^{zz}\delta^A_B - g^{Az}\delta^z_B)\}. \quad (4.80)$$

This energy-momentum tensor describes the matter content (fluid) of a thin disk located on $z = 0$. In first approximation one can consider that the particles of such a fluid move along geodesics. In particular, we can consider particles moving along circular geodesics whose tangential velocities give us the rotation curves. Calculating the extradimensional components of the energy-momentum tensor, we find that $[T^{x^i}_{x^i}] = -[T^{x^{i+1}}_{x^{i+1}}] = \frac{1}{16\pi}g^{zz}b^{x^i}_{x^i}$. If D is even, then the sum of extradimensional pressures cancel and the total pressure depends only of $4D$ components. Thus, the density profiles appear to be the same as in the $4D$ case and we find that the extra dimensions do not affect the density and the azimuthal and radial pressures: this is a striking result because we only need the direct observed density profiles to explain the content of the disk. Density profiles for $4D$ thin disks can be seen e.g. in Bičák *et al.* [45]. Other important arguments can be found in literature in favor of an even D (e.g. as has been emphasized by different authors, the Huygens principle does not hold for odd D [46]).

As the simplest example, we present in what follows the case for a 6D disk. The metric for an axially symmetric 6D space-time can be written in quasi-cylindrical coordinates as

$$\mathrm{d}s^2 = -\mathrm{e}^{-\phi}\mathrm{d}t^2 + \chi^2\mathrm{e}^{\phi}\mathrm{d}\varphi^2 + \psi\mathrm{e}^{\nu}\mathrm{d}x^2 + \mathrm{e}^{-\nu}\mathrm{d}y^2 + f(\mathrm{d}r^2 + \mathrm{d}z^2), \quad (4.81)$$

where $\phi = \phi(r,z)$, $f = f(r,z)$, $\chi = \chi(r,z)$, $\psi = \psi(r,z)$ and x and y are the extra dimensional coordinates; we do $G = 1$ and $c = 1$. The vacuum Einstein equations $R_{AB} = 0$, $(A, B = 0, 1, ..., 5)$ reduce to

$$(\chi\sqrt{\psi})_{,rr} + (\chi\sqrt{\psi})_{,zz} = 0, \quad (4.82)$$

$$\nu_{,rr} + \frac{\nu_{,r}(\chi\sqrt{\psi})_{,r}}{\chi\sqrt{\psi}} + \frac{\nu_{,z}(\chi\sqrt{\psi})_{,z}}{\chi\sqrt{\psi}} + \nu_{,zz} = 0, \quad (4.83)$$

$$\phi_{,rr} + \frac{\phi_{,r}(\chi\sqrt{\psi})_{,r}}{\chi\sqrt{\psi}} + \frac{\phi_{,z}(\chi\sqrt{\psi})_{,z}}{\chi\sqrt{\psi}} + \phi_{,zz} = 0. \quad (4.84)$$

where $(\ )_{,a} = \partial/\partial x^a$. It can be shown that, without lossing generality, one can choose $\chi\sqrt{\psi} = r$. We do $\psi = 1$ and $\chi = r$, that yields the Einstein equation for $f(r,z)$,

$$\ln f[\phi,\nu] = \frac{1}{2}\int r\{[\phi^2_{,r} - \phi^2_{,z} + 2\phi_{,r}/r + \nu^2_{,r} - \nu^2_{,z}]\mathrm{d}r + [2\phi_{,r}\phi_{,z} + 2\phi_{,z}/r + 2\nu_{,r}\nu_{,z}]\mathrm{d}z\}. \quad (4.85)$$

Eqs.(4.82)–(4.85) form the complete set of vacuum Einstein equations for the metric (4.81).

To obtain a solution of (4.82)–(4.85) which represents a thin disk located on $z = 0$, we assume the functions of the metric f and ϕ are continuous along the disk, in particular on the surface $z = 0$, but with discontinuous first derivatives on that surface. We introduce these discontinuities by doing the replacement $z \to |z| + a$, where a is a constant. The distributional energy-momentum tensor reads

$$[T^t{}_t] = \frac{1}{16\pi}\{-b^{zz} + g^{zz}(b^r{}_r + b^z{}_z + b^\varphi{}_\varphi + b^x{}_x + b^y{}_y)\}, \quad (4.86)$$

$$[T^r{}_r] = \frac{1}{16\pi}\{-b^{zz} + g^{zz}(b^t{}_t + b^z{}_z + b^\varphi{}_\varphi + b^x{}_x + b^y{}_y)\}, \quad (4.87)$$

$$[T^z{}_z] = 0, \quad (4.88)$$

$$[T^\varphi{}_\varphi] = \frac{1}{16\pi}\{-b^{zz} + g^{zz}(b^t{}_t + b^r{}_r + b^z{}_z + b^x{}_x + b^y{}_y)\}, \quad (4.89)$$

$$[T^x{}_x] = \frac{1}{16\pi}\{-b^{zz} + g^{zz}(b^t{}_t + b^r{}_r + b^z{}_z + b^\varphi{}_\varphi + b^y{}_y)\}, \quad (4.90)$$

$$[T^y{}_y] = \frac{1}{16\pi}\{-b^{zz} + g^{zz}(b^t{}_t + b^r{}_r + b^z{}_z + b^\varphi{}_\varphi + b^x{}_x)\}. \quad (4.91)$$

And the discontinuities of first derivatives of the metric $b_{AB} = (g^{+}_{AB,z} - g^{-}_{AB,z})|_{z=0}$ yields

$$b_{tt} = 2\mathrm{e}^{-\phi}\phi_{,z}|_{z=0}, \quad (4.92)$$

$$b_{rr} = 2f_{,z}|_{z=0}, \quad (4.93)$$

$$b_{zz} = b_{rr}, \quad (4.94)$$

$$b_{\phi\phi} = 2r^2\mathrm{e}^{\phi}\phi_{,z}|_{z=0}, \quad (4.95)$$

$$b_{xx} = 2\mathrm{e}^{\nu}\nu_{,z}|_{z=0}, \quad (4.96)$$

$$b_{yy} = -2\mathrm{e}^{-\nu}\nu_{,z}|_{z=0}. \quad (4.97)$$

Writing a new vielbein that includes extra dimensions as

$$e_{(t)}{}^A = \left(\frac{1}{\sqrt{-g_{tt}}},0,0,0,0,0\right), e_{(r)}{}^A = \left(0,\frac{1}{\sqrt{g_{rr}}},0,0,0,0\right), e_{(\varphi)}{}^A = \left(0,0,\frac{1}{\sqrt{g_{\varphi\varphi}}},0,0,0\right),$$
$$e_{(z)}{}^A = \left(0,0,0,\frac{1}{\sqrt{g_{zz}}},0,0\right), e_{(x)}{}^A = \left(0,0,0,0,\frac{1}{\sqrt{g_{xx}}},0\right), e_{(y)}{}^A = \left(0,0,0,0,0,\frac{1}{\sqrt{g_{yy}}}\right),$$

one can write down the energy-momentum tensor (2.24) as

$$\begin{aligned}[T^A{}_B] = &- \epsilon e_{(t)}{}^A e_{(t)}{}^B + p_r e_{(r)}{}^A e_{(r)}{}^B \\ &+ p_\varphi e_{(\varphi)}{}^A e_{(\varphi)}{}^B + p_z e_{(z)}{}^A e_{(z)}{}^B \\ &+ p_x e_{(x)}{}^A e_{(x)}{}^B + p_y e_{(y)}{}^A e_{(y)}{}^B, \end{aligned} \tag{4.98}$$

yielding the energy density and pressures as

$$\epsilon = -[T^t{}_t] = -\frac{f_{,z}}{8\pi f^2}|_{z=0} \tag{4.99}$$

$$p_\varphi = [T^\varphi{}_\varphi] = -\frac{\phi_{,z}}{8\pi f}|_{z=0} \tag{4.100}$$

$$p_r = [T^r{}_r] = 0, \tag{4.101}$$

$$p_z = [T^z{}_z] = 0. \tag{4.102}$$

$$p_x + p_y = 0. \tag{4.103}$$

A second step to construct the disk is to choose two solutions of Laplace equations (4.83) and (4.84) for the functions ϕ and ν. The D-dimensional Schwarzschild solution has isometry group $\mathbf{R} \times O(D-1)$. To write it in Weyl form, $D-2$ orthogonal commuting Killing vector fields are required. For the Schwarzschild solution, this occurs only for $D = 4, 5$. Hence only the four and five-dimensional Schwarzschild solutions can be written in Weyl form. For $D > 5$, the D-dimensional Schwarzschild solution is not a generalized Weyl solution. However, the geometry obtained by taking products of the $D = 4$ or $D = 5$ Schwarzschild solution with asymptotically flat space are easily seen to be Weyl solutions.

We take the Newtonian potential associated to a rod of constant density [47],

$$\phi(r,z) = \ln\frac{R_1 + R_2 - 2}{R_1 + R_2 + 2}, \tag{4.104}$$

where $R_1 = \sqrt{\tilde{r}^2 + (z-1)^2}$ e $R_2 = \sqrt{\tilde{r}^2 + (z+1)^2}$, and $\tilde{r}$ is the radial coordinate normalized by mass, $\tilde{r} = r/m$. The length of the rod is $L = 2m$. We also use a Chazy-Curzon solution for the ν function ($\nu(r,z) = [r^2 + (|z| + a)^2]^{-1/2}$), which is asymptotically flat, resulting that our disk is in fact a Weyl solution.

V. ROTATION CURVES

From Eq. (4.81) we have the first integral of motion,

$$-\mathrm{e}^{-\phi}\dot{t}^2 + f(\dot{r}^2+\dot{z}^2) + r^2\mathrm{e}^{\phi}\dot{\varphi}^2 + \mathrm{e}^{\nu}\dot{x}^2 + \mathrm{e}^{-\nu}\dot{y}^2 = 1, \tag{5.105}$$

where $\dot{x}^A = \mathrm{d}x^A/\mathrm{d}s$. Assuming $\dot{r} = 0$ and $\dot{z} = 0$ (particles with no radial motion and confined on $z = 0$), Eq.(5.105) reads

$$-\mathrm{e}^{-\phi}\dot{t}^2 + r^2\mathrm{e}^{\phi}\dot{\varphi}^2 + \mathrm{e}^{\nu}\dot{x}^2 + \mathrm{e}^{-\nu}\dot{y}^2 = 1. \tag{5.106}$$

By Eqs. (4.99)–(4.103), one evidently can see that extra dimensions pressures do not contribute to total disk pressures. Immediately one is tempted to argue that this is the same to say that extra dimensions do not contribute to the density profiles. However, Eq. (4.85) shows that $f(r,z)$ contains extra dimensional field components and therefore *a priori* the rates $\dot{x}$ and $\dot{y}$ must be considered as non-null incognites. On this fashion, the geodesic equations on the disk reduce to

$$\mathrm{e}^{\nu}\dot{x} = C_x, \quad \mathrm{e}^{-\nu}\dot{y} = C_y, \tag{5.107}$$

$$\begin{aligned} &(\mathrm{e}^{-\phi})_{,r}\dot{t}^2 - (r^2\mathrm{e}^{\phi})_{,r}\dot{\varphi}^2 \\ = \; &C_x^2(\mathrm{e}^{-\nu})_{,r} + C_y^2(\mathrm{e}^{\nu})_{,r}, \end{aligned} \tag{5.108}$$

where C_x and C_y are integration constants.

Eqs. (5.106) and (5.108) form a system of equations for $\dot{\varphi}^2$ and $\dot{t}^2$. From these equations we find the rotation curves V_C,

$$V_C = \sqrt{-\frac{g_{\varphi\varphi}}{g_{tt}}}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \sqrt{-\frac{g_{\varphi\varphi}}{g_{tt}}\frac{\dot{\varphi}^2}{\dot{t}^2}}, \tag{5.109}$$

reduce to

$$V_C = \sqrt{\frac{F(r)\nu_{,r} + G(r)\phi_{,r}}{F(r)\nu_{,r} - G(r)(2/r + \phi_{,r})}}, \tag{5.110}$$

where $F(r) = -C_x^2\mathrm{e}^{-\nu} + C_y^2\mathrm{e}^{\nu}$ and $G(r) = 1 - C_x^2\mathrm{e}^{-\nu} - C_y^2\mathrm{e}^{\nu}$. Note that when $C_x = C_y = 0$ (no extra dimensions), we have $V_C = \sqrt{\phi_{,r}/(2/r + \phi_{,r})}$ that is the known formula for circular orbits in 4D Weyl geometry, Eq. (2.39). The boundary conditions in manner to determine values for C_x and C_y come from calculations of the stability of the disk, a subject studied in what follows.

VI. THE STABILITY

A. Rayleigh criterion

The stability of the circular orbits in the disk plane can be studied using an extension of the Rayleigh stability criterion [48]. We have stability when $h\frac{\mathrm{d}h}{\mathrm{d}r} > 0$, where h is the specific angular momentum of a particle in the disk plane ($h = g_{\varphi\varphi}\dot{\varphi}$),

$$h = r^2 \mathrm{e}^{\phi} \sqrt{\frac{(1 - C_x^2 \mathrm{e}^{-\nu} - C_y^2 \mathrm{e}^{\nu})\phi_{,r} + (C_x^2 \mathrm{e}^{-\nu} - C_y^2 \mathrm{e}^{\nu})\nu_{,r}}{2r^2 \mathrm{e}^{\phi}(1/r + \phi_{,r})}}. \tag{6.111}$$

We find that for different values of C_x and C_y less than unity stability is reached when $a > 1$. For small values of a (highly relativistic disks) we have a small zone of instability, typically around $r = 3m$. In Fig. 4 we show the stability of the disk by the criterion presented above.

B. Perturbative method

The stability of circular orbits in the disk plane can be also studied using a perturbative method where we assume that the disk particles are describing equatorial circular geodesics in stationary axisymmetric fields. The perturbation of the geodesic equation $\ddot{x}^A + \Gamma^A{}_{BC}\dot{x}^B\dot{x}^C = 0$ is done performing the transformation $x^A \to x^A + \Delta^A$ – where $\Delta^A = (\delta t, \delta r, \delta\varphi, \delta z, \delta x, \delta y)$ are infinitesimal elements. Therefore, equations for the perturbations are,

$$\ddot{\Delta}^A + 2\Gamma^A{}_{BC}\dot{x}^B\dot{\Delta}^C + \Gamma^A{}_{BC,D}\Delta^D\dot{x}^B\dot{x}^C = 0, \tag{6.112}$$

where $\Gamma^A{}_{BC}$ are the Christoffel symbols and $\dot{x}^A$ are proper time derivatives $\mathrm{d}x^A/\mathrm{d}s$ and can be written for a circular orbital motion as

$$\dot{x}^A = (u^t, 0, 0, u^t\Omega, C_x, C_y), \tag{6.113}$$

where $u^t\Omega = V_C$. Note that (6.112) is equivalent to the usual deviation equation [49]. Assuming only horizontal oscillations in the 4D part of the disk ($\delta z = 0$), we get

$$\Delta^A = (\delta t, \delta r, 0, 0, \delta\varphi, 0, 0). \tag{6.114}$$

Let x^A be an equatorial circular geodesic in a stationary axisymmetric space-time (4.81), i.e., the worldline $x^A = (t, r = \text{const}, \varphi = \text{const} + \Omega t, z = 0, x = \text{const}, y = \text{const})$. Substituting the four velocity (6.113) and demanding that $g_{AB,z}$ (but not $g_{AB,zz}$) vanishes in the equatorial plane, the components of Eq.(6.112) for horizontal oscillations read:

$$(\ddot{\delta t}) + 2\Gamma^t{}_{tr}u^t(\dot{\delta r}) = 0, \tag{6.115}$$

$$(\ddot{\delta r}) + 2\Gamma^r{}_{tt} + 2\Gamma^r{}_{\varphi\varphi}u^t\Omega(\dot{\delta\varphi}) + [(\Gamma^r{}_{tt,r} + \Gamma^r{}_{\varphi\varphi,r}\Omega^2)(u^t)^2 + \Gamma^r{}_{xx,r}C_x^2 + \Gamma^r{}_{yy,r}C_y^2]\delta r = 0, \tag{6.116}$$

$$(\ddot{\delta\varphi}) + 2\Gamma^\varphi{}_{\varphi r}\Omega u^t(\dot{\delta r}) = 0. \tag{6.117}$$

Suppose that the solutions for δt, δr and $\delta\varphi$ have a form of harmonic oscillations, $\sim e^{iKs}$, with a common proper angular frequency K. The condition for solvability of Eqs. (6.115)-(6.117) is then

$$\det\begin{pmatrix} -K^2 & 2iK\Gamma^t{}_{tr}u^t & 0 \\ 2iK\Gamma^r{}_{tt}u^t & -K^2 + \Gamma^r{}_{AB,r}u^Au^B & 2iK\Gamma^r{}_{\varphi\varphi}u^t\Omega \\ 0 & 2iK\Gamma^\varphi{}_{\varphi r}u^t\Omega & -K^2 \end{pmatrix} = 0, \tag{6.118}$$

where $\Gamma^r{}_{AB,r}u^Au^B = (\Gamma^r{}_{tt,r} + \Gamma^r{}_{\varphi\varphi,r}\Omega^2)(u^t)^2 + \Gamma^r{}_{xx,r}C_x^2 + \Gamma^r{}_{yy,r}C_y^2$. From the non-trivial solution of this equation we derive the oscillation frequency with respect to infinity $\kappa = K/u^t$, referred in literature as the epicyclic frequency [50], as

$$\kappa^2 = \frac{Z(r)}{2 + \phi_{,r} - P(r)}[\phi_{,rr} + r\phi_r^3 + 3\phi_{,r}/r + 3\phi_r^2 + Q(r)], \tag{6.119}$$

where $Z(r) = -e^{-\phi}/f$, the metric is that given by Eq.(4.81) and $P(r)$ and $Q(r)$ are terms related to extradimensional imprints:

$$P(r) = M(r)[2 + \phi_{,r}r] + F(r)r\nu_{,r}, \tag{6.120}$$

$$Q(r) = P(r)[H(r)/r + 0.5\phi_{,r}f_{,r}/f - \phi_{,r}^2/2 - \phi_{,rr}/2] - H(r)\phi_{,r} - 2H(r)/r - N(r)/r, \tag{6.121}$$

$$H(r) = \frac{e^{\nu-\phi}}{2r}C_x^2[\nu_{,r}f_{,r}/f - \nu_{,r}^2 - \nu_{,rr}] + \frac{e^{-\nu-\phi}}{2r}C_y^2[-\nu_{,r}f_{,r}/f - \nu_{,r}^2 + \nu_{,rr}] \tag{6.122}$$

$$N(r) = F(r)[-3\nu_{,r} - 2r\phi_{,r}\nu_{,r} - 0.5r^2\phi_{,r}^2\nu_{,r} + 0.5r^2\phi_{,rr}] + M(r)[3\phi_{,r} + 2r\phi_{,r}^2 + 0.5r^2\phi_{,r}^3 - 0.5r^2\phi_{,rr}\phi_{,r} + r\phi_{,r}f_{,r}/f + 0.5r^2\phi_{,r}^2 f_{,r}/f], \tag{6.123}$$

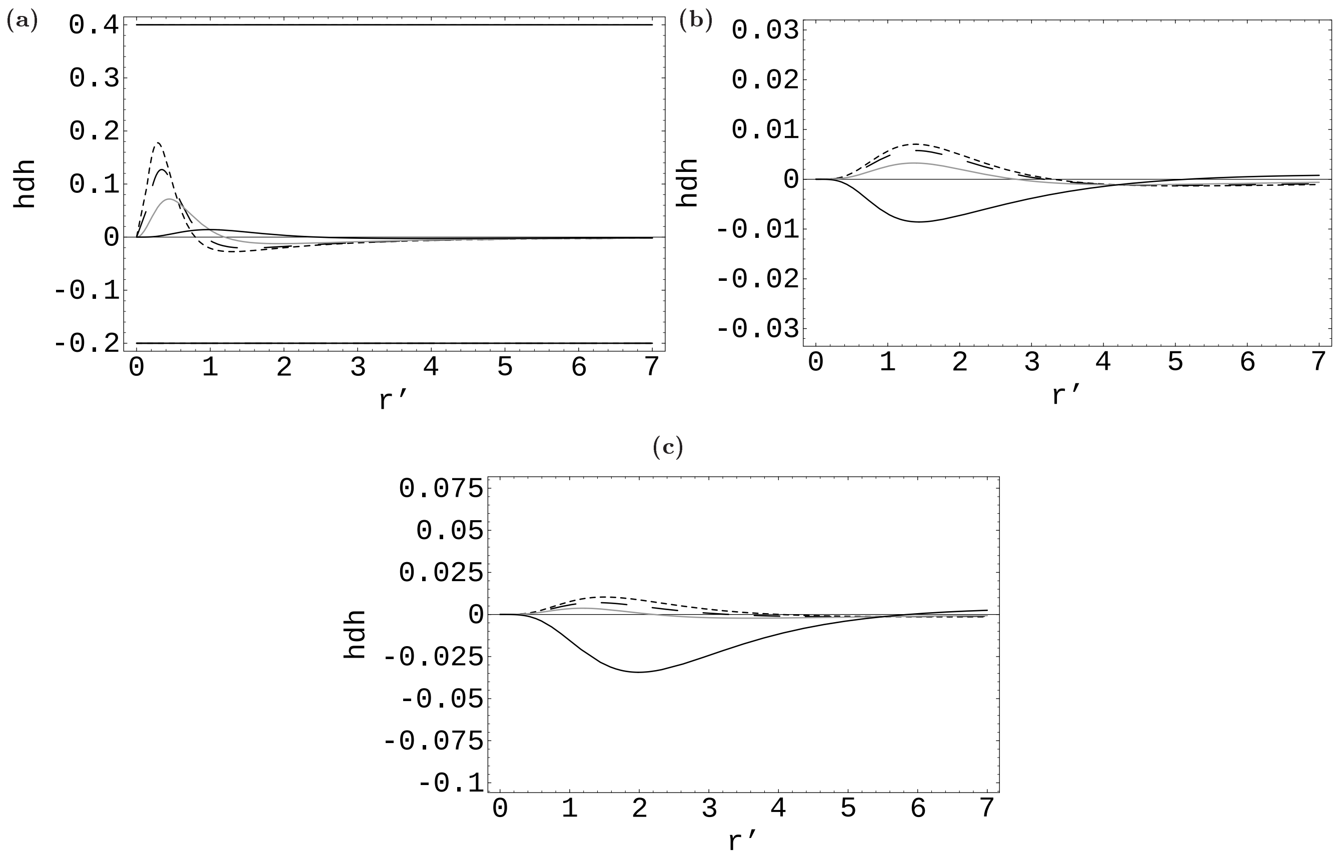


FIG. 4: Stability of disks by the modified Rayleigh criterion. In **(a)** we show that stable disk cut parameters a are obtained only for $a > 1$ (the full line), where $h\frac{\mathrm{d}h}{\mathrm{d}r} > 0$. **(b)** In the region of interest, stable disks occur for extradimensional parameters $C_x = 0.8$ (dotted line), $C_x = 0.85$ (dashed line) and $C_x = 0.9$ (gray line). For $C_x > 0.95$, $h\frac{\mathrm{d}h}{\mathrm{d}r} < 0$, and the disk becomes unstable (full line). **(c)** In the region of interest, stable disks occur for extradimensional parameters $C_y = 0.1$ (dotted line), $C_y = 0.2$ (dashed line) and $C_y = 0.3$ (gray line). For $C_y > 0.5$, $h\frac{\mathrm{d}h}{\mathrm{d}r} < 0$, and the disk becomes unstable (full line).

and where $M(r) = C_x^2 e^{-\nu} + C_y^2 e^{\nu}$ and $F(r) = -C_x^2 e^{-\nu} + C_y^2 e^{\nu}$. When $C_x = C_y = 0 \Rightarrow P(r) = Q(r) = 0$ (no extra dimensions), Eq.(6.119) becomes the known formula for oscillations in Weyl geometry. Graphics for the squared epicyclic frequency is showed in Fig. 5. The configuration is stable only when $\kappa^2 > 0$. Our results show that the integration constants C_x and C_y have a very restricted range of stable values. In Table I it is possible to see the intervals for C_x and C_y where the disk is stable (here we fixed $a = 1.5$, a stable disk parameter according to the Rayleigh criterion; any values $a > 1$ are able to produce stable disks).

A such stability study is fundamental to discuss what are the exact values to be used for the extradimensional parameters C_x and C_y. In general, extra dimensions contribute to destabilize the disk, but it is possible to derive a semi-phenomenological model where we have a very short range of stable values that can be used to fit rotation curves of galaxies or gravitational lensing of dark halos in a UED background (see Table I). Those stable values can astrophysically constrain two-UED models. A more detailed discussion on the present perturbative method is presented in [51].

VII. STABLE ROTATION CURVES FOR A $6D$ DISK

The stability study of the previous section is fundamental to discuss what are the exact values to be used for the extradimensional parameters C_x and C_y. In this sense, it is possible to derive a semi-phenomenological model where we have a very short range of stable values that can be used to compare to rotation curves of galax-

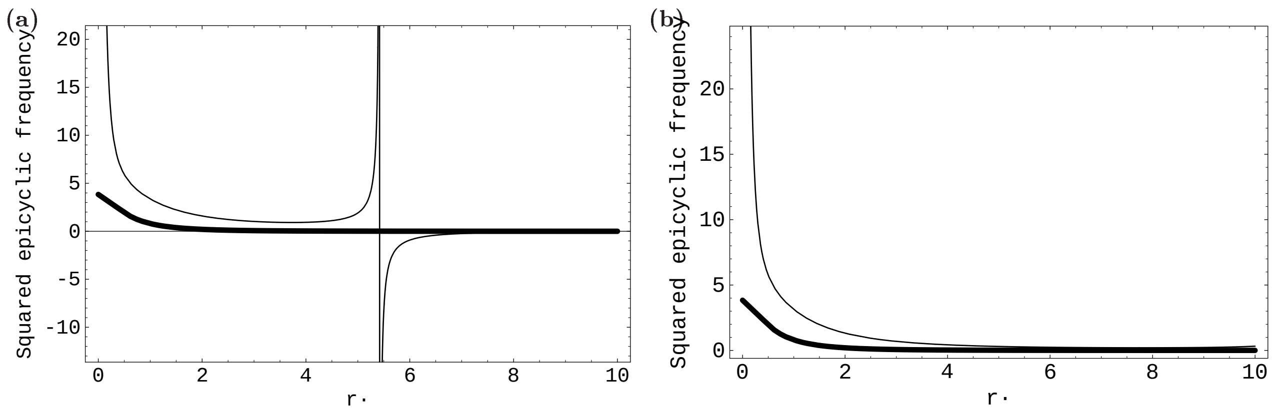


FIG. 5: **(a)** Epicyclic frequency versus the normalized radius $r' = r/m$ for a configuration where $C_x = 0.1$ and $C_y = 0.85$. The stability is achieved for $0 < r' < 5.5$, when $\kappa^2 > 0$; a great number of rotation curves of spiral galaxies remains exactly into this region. The bold curve is the result for a 4D thin disk configuration. **(b)** Epicyclic frequency versus the normalized radius for a configuration where $C_x = 0.1$ and $C_y = 0.9$. Here the stability is achieved for $0 < r' < 15$, when $\kappa^2 > 0$. Here $r' = r/m$.

TABLE I: Stable values for C_x and C_y

Values for C_x and C_y	Region where the disk is stable
$C_x = 0$, $C_y = 0$ (Newtonian)	**All** r'
$0 < C_x < 0.2$, $0 < C_y < 0.4$	**All** r'
$0 < C_x < 0.2$, $C_y = 0.5$	$0 < r' \lesssim 0.4$
$0 < C_x < 0.2$, $C_y = 0.7$	$0 < r' \lesssim 1.4$
$0 < C_x < 0.2$, $C_y = 0.75$	$0 < r' \lesssim 2.6$
$0 < C_x < 0.2$, $C_y = 0.8$	$0 < r' \lesssim 4$
$0 < C_x < 0.2$, $C_y = 0.85$	$0 < r' \lesssim 5.5$ (fit DM halos)
$0 < C_x < 0.2$, $C_y = 0.9$	$0 < r' \lesssim 15$ (fit DM halos)
$0 < C_x < 0.2$, $C_y > 0.95$	**Unstable disk**

ies. In Fig. 6a we show the rotation curves for some values of C_x and C_y, where we are not pondering yet the stable cases. In 6b we present only the stable curves, for various values of the cut parameter a. As discussed before, the stability is only achieved for $0 < C_x < 0.4$ and $0 < C_y < 0.95$ (Rayleigh criterion) or the values specified in Table I (what restringes even more the range for C_x and confirms the Rayleigh criterion for C_y). At the same time those values also prevent superluminal behavior for the particles. When $C_x = C_y = 0$ we have the usual $4D$ general relativistic profile that is quite similar to a typical Newtonian one. On the other hand, the "flattening" of galactic rotation curves is an effect observed to occur in the galactic area that contains most of the baryonic (visible) matter. Beyond this area it becomes extremely difficult to ascertain the behavior of rotation profiles, so a model that just keeps these rotation profiles flat may be incorrect at larger cosmological scales. In Fig. 6c we show that the curves derived are asymptotically flat and in cosmological scales the curves tend to zero. The density profiles as discussed previously are practically the same as in the $4D$ case and we find that the extra dimensions do not affect the density and the azimuthal and radial pressures. These profiles can be seen in Fig. 2 and in Bičák et al. [18]. In Fig. 7 we show a comparison between the profiles derived with and without extradimensional imprints, showing that the disagreement can be easily neglected, although it is exactly due to this tiny disagreement that KK modes affect rotation curves.

Taking into account only the stable curves, in Fig. 8 we compare with some optically observed rotation curves of spiral galaxies. Here we are not doing a composition of a halo dark matter velocities plus the disk gas and the velocities of stars. What is happening is that the clean stable general relativistic disk geodesics are simply fitting the region of interest. The not surprising *ad hoc* adjustment of C_x and C_y actually could tell nothing about the astrophysical role of the KK modes in the model. However the calculation of stable disks brings over with it realizable values for C_x and C_y, what makes possible to visualize a minimum representation of a real disk galaxy. Those values produce the full line curves of Fig. 6b, fitting with great precision the region of interest (the plateau anomaly after $r' \sim 2$). We also compare to some phenomenological models used in Astrophysics as Navarro, Frenk & White [56] and the Courteau fit [57]. The observed data is taken from [52, 53, 54], but for an alternative data source see [4, 55], where our model is also successful. Another important statement is that the present model recovers the Newtonian profile for $r' \to \infty$ (where the asymptotic function $\nu \to 0$, and the Newtonian limit is reached).

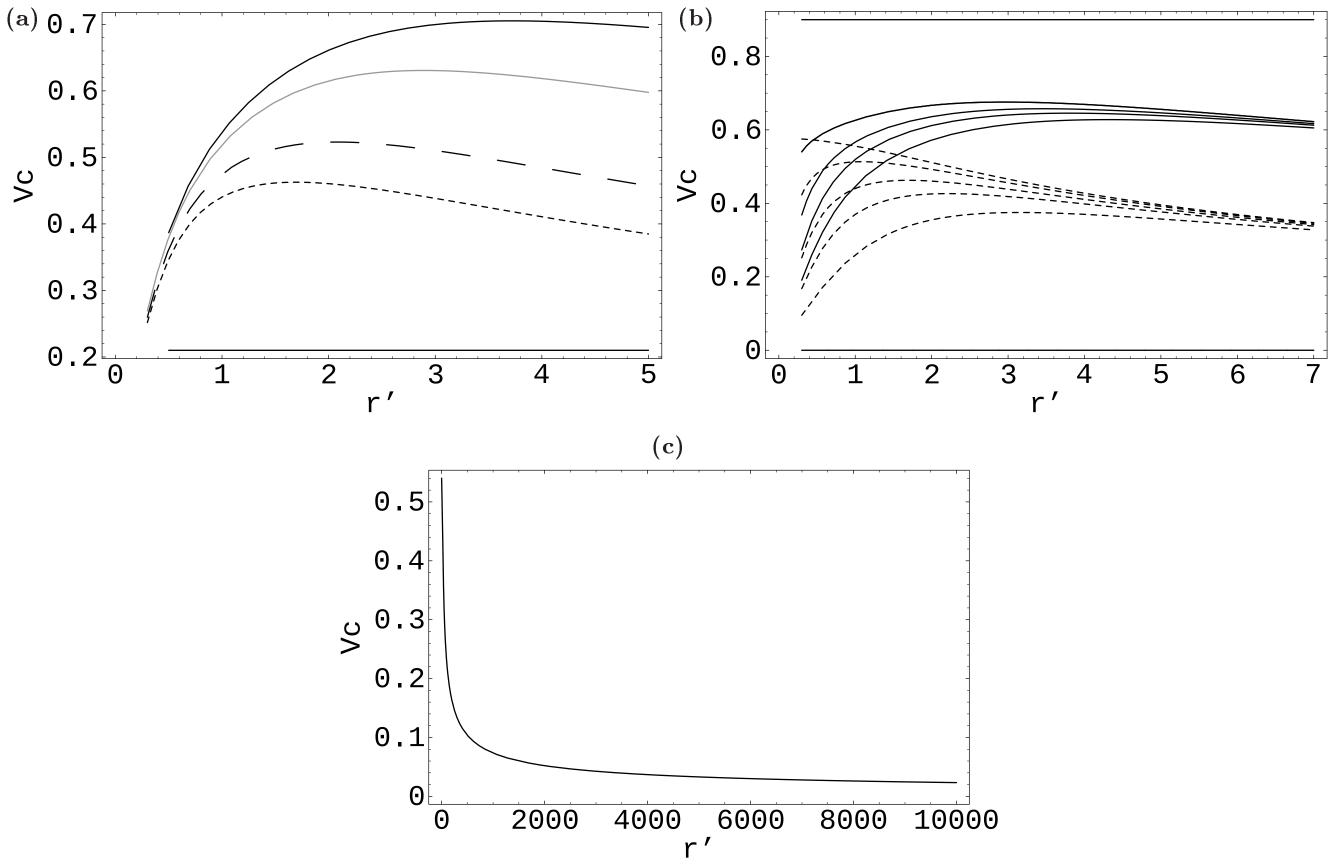


FIG. 6: **(a)** Disk rotation curves with extra dimensional parameters $C_x = 0$, $C_y = 0$, i.e., usual Newtonian profile (dotted line, stable), ; $C_x = 0.1$, $C_y = 0.7$ (dashed line, unstable); $C_x = 0.1$, $C_y = 0.85$ (gray line, stable); $C_x = 0.1$, $C_y = 0.9$ (full line, stable). We take $r' = r/m$. **(b)** Only the stable curves, for various values of the cut parameter a; here a is varying from $a = 1$ to $a = 2.5$. The dotted ones are the Newtonian-like (where there are no extra dimensions), and the full ones are the stable curves derived admitting two extra dimensions. In **(c)** we show that the curves derived are asymptotically flat, and in large scale they tend to zero.

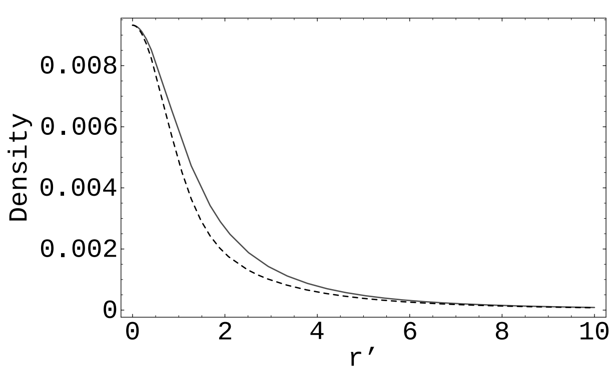


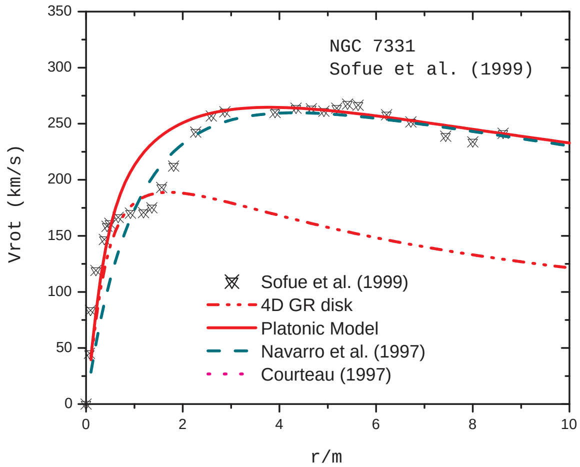
FIG. 7: Comparison of the density profiles of a disk considering only $4D$ dimensions (full line) and what is derived by the present example that considers $6D$ (dotted line). It is important to point that it is exactly due to this tiny disagreement that KK modes affect rotation curves. Here a stable example with $a = 1.5$, for $m = 1$.

It is important to remember that this is not a pure phenomenological model, where we would be interested in take a complete fit of observational curve. Here the interest is to achieve a reasonable explanation for the plateau anomaly using some of the stable calculated parameters. In this sense, the model could be considered as a semi-phenomenological model.

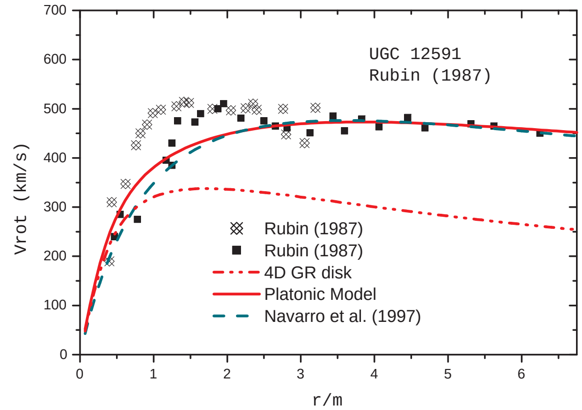
VIII. GRAVITATIONAL LENSING

Here, using the simple example for $6D$, we show that in the low acceleration regime, a universe endowed with UED in a Platonic regime predicts gravitational lensing of the correct magnitude to explain the observations of intergalactic lensing. As in nature many elliptical galaxies and galaxy clusters are well modeled as spherically symmetric, we do our calculation for a spherically sym-

(a)



(b)



(c)

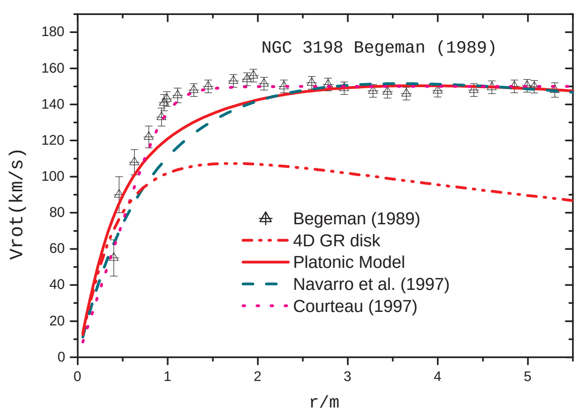


FIG. 8: **(a)** Different models (our model and [56]) for the rotation curves of the NGC 7331 spiral galaxy. Observational data from Sofue et al. (1999) [52]. **(b)** Different models (our model and [56]) for rotation curves of the high speed rotation spiral galaxy UGC 12591. Observational data from Rubin (1987) [53]. **(c)** Different models (our model and [56, 57]) for the rotation curves of the NGC 3198 spiral galaxy. Observational data from Begeman (1999) [55]. The model fits with great precision the region of interest – the plateau anomaly after $r/m \sim 3$ in **(a)** and after $r/m \sim 2$ in **(b)** and **(c)**. The stable parameters used are $C_x = 0.15$ and $C_y = 0.88$ for **(a)** and **(b)**, and $C_x = 0.2$ and $C_y = 0.85$ for **(c)**.

metric system plus two UED. We adopt the metric

$$\mathrm{d}s^2 = -e^{\Phi}\mathrm{d}t^2 \;+\; e^{\Lambda}[\mathrm{d}R^2 + R^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2) \\ +\; \mathrm{d}x^2 + \mathrm{d}y^2], \tag{8.124}$$

where, as before, x and y are the flat extradimensional coordinates. Φ and Λ are functions only of R. Consider a light ray which propagates in the equatorial plane of the metric. The velocity $\dot{x}^A$ of the ray must satisfy

$$-\,e^{\Phi}\,\dot{t}^2 + e^{\Lambda}(\dot{R}^2 + R^2\dot{\varphi}^2 + \dot{x}^2 + \dot{y}^2) = 0. \tag{8.125}$$

From the stationarity of the metric follows the conservation law $e^{\Phi}\dot{t} = E$ where E is a constant characteristic of the ray. From spherical symmetry it follows that $e^{\Lambda}R^2\dot{\varphi} = L$ where L is another constant property of the ray. Let us write $\dot{R} = (\mathrm{d}R/\mathrm{d}\varphi)\dot{\varphi}$, $\dot{x} = (\mathrm{d}x/\mathrm{d}\varphi)\dot{\varphi}$ and $\dot{y} = (\mathrm{d}y/\mathrm{d}\varphi)\dot{\varphi}$. Now eliminating $\dot{t}$ and $\dot{\varphi}$ from Eq. (8.125) in favor of E and L, and dividing by E^2 yields

$$-\,e^{-\Phi} \;+\; (b/R)^2 e^{-\Lambda}\{R^{-2}[(\mathrm{d}R/\mathrm{d}\varphi)^2 + (\mathrm{d}x/\mathrm{d}\varphi)^2 \\ +\; (\mathrm{d}y/\mathrm{d}\varphi)^2] + 1\} = 0, \tag{8.126}$$

where $b \equiv L/E$. By going to infinity where the metric factors approach unity one sees that b is just the impact parameter of the ray with respect to the matter $4D$ distribution center at $R = 0$. Rearranging the last equation we obtain the quadrature

$$\varphi = [1 + R_x^2 + R_y^2]^{1/2} \int^R \left[e^{\Lambda - \Phi} \left(\frac{R}{b} \right)^2 - 1 \right]^{-1/2} \frac{\mathrm{d}R}{R}, \tag{8.127}$$

where $R_x^2 = (\mathrm{d}x/\mathrm{d}R)^2$ and $R_y^2 = (\mathrm{d}y/\mathrm{d}R)^2$ are the rates where the KK imprints are distributed along the $4D$ galaxy cluster. At this point, where the physical metric exactly flat, this relation would describe a line with φ varying from 0 to π as R decreased from infinity to its value R_{turn} at the turning point, and then returned to infinity. Therefore the deflection of the ray due to gravity is

$$\Delta\varphi = [1 + R_x^2 + R_y^2]^{1/2} \left\{ 2 \int_{R_{turn}}^{\infty} \left[e^{\Lambda - \Phi} \left(\frac{R}{b} \right)^2 - 1 \right]^{-1/2} \frac{\mathrm{d}R}{R} - \pi \right\}. \tag{8.128}$$

To shed some light in this last integral, one can benefit of the weakness of extragalactic fields which grant that Λ and Φ are all small compared to unity. As consequence the above result is closely approximated by

$$\Delta\varphi = [1 + R_x^2 + R_y^2]^{1/2} \left\{ -4 \frac{\partial}{\partial\alpha} \int_{R_{turn}}^{\infty} \left[(1 + \Lambda - \Phi) \left(\frac{R}{b} \right)^2 - \alpha \right]^{1/2} \frac{\mathrm{d}R}{R} \Big|_{\alpha=1} - \pi \right\}. \tag{8.129}$$

The rewriting in terms of an α derivative allows us to Taylor expand the radical in the small quantity $\Lambda - \Phi$ without incurring a divergence of the integral at its lower limit. The zeroth order of the expansion yields a well known integral which cancels the π. Thus, to first order in small quantities

$$\Delta\varphi = -\frac{2\,[1 + R_x^2 + R_y^2]^{1/2}}{b} \frac{\partial}{\partial\alpha} \int_{b\sqrt{\alpha}}^{\infty} \frac{(\Lambda - \Phi) R \mathrm{d}R}{(R^2 - \alpha b^2)^{1/2}} \Big|_{\alpha=1}. \tag{8.130}$$

And integrating by parts:

$$\Delta\varphi = -\frac{2\,[1 + R_x^2 + R_y^2]^{1/2}}{b} \frac{\partial}{\partial\alpha} \Big[\lim_{R\to\infty} (\Lambda - \Phi)(R^2 - \alpha b^2)^{1/2} - \int_{b\sqrt{\alpha}}^{\infty} (\Lambda_{,R} - \Phi_{,R})(R^2 - \alpha b^2)^{1/2} \mathrm{d}R \Big] \Big|_{\alpha=1} \tag{8.131}$$

Since Φ and Λ decrease asymptotically as R^{-1}, the integrated term, being α independent, contributes nothing. Carrying out the α derivative, and introducing the usual Cartesian u coordinate along the initial ray by $u \equiv \pm(R^2 - b^2)^{1/2}$, we have

$$\Delta\varphi = \frac{b\,[1 + R_x^2 + R_y^2]^{1/2}}{2} \int_{-\infty}^{\infty} \frac{\Lambda_{,R} - \Phi_{,R}}{R} \mathrm{d}u. \tag{8.132}$$

A factor 1/2 appears because we have included the integral in Eq. (8.131) twice, once with R decreasing to, and once with R increasing from b. The integral is now performed over an infinite straight line following the original ray.

Fig. 9 shows that the new ray deflection using the above calculated model, for a range of projected rates R_x and R_y, is bigger than the deflection produced by a ray passing through a cluster calculated only with $4D$ general relativity (GR). The difference between a $4D$ GR with dark matter and the above calculation (where DM comes from Platonic Kaluza-Klein modes) is that in a conventional dark matter scenario one would compute Φ and Λ from Einstein equations including dark matter as source, whereas in the Platonic Model one has an aditional term and computes Λ and Φ on the basis of the visible matter alone.

Solving numerically the Einstein equations of the spherical metric and by a numerical integration of Eq. (8.132), it is possible to obtain a comparison between a galactic cluster with and without extra dimensions. In Fig. 10 we demonstrate this, using the same values for C_x and C_y calculated in Sec. VI. In our model, the isotropic KK matter constitutes $(79.5 \pm 3.3)\%$ of the total matter in a spherically symmetric galactic cluster. This is very similar to what happens accordingly to observations. It is

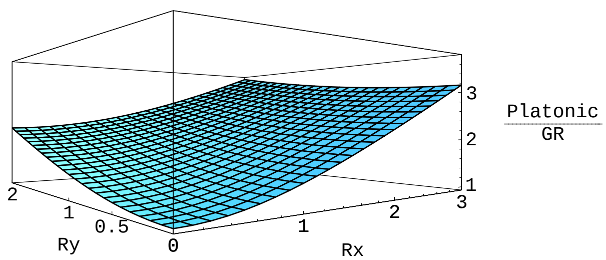


FIG. 9: The new ray deflection using the above calculated model (Platonic), for a range of projected rates R_x and R_y, is bigger than the deflection produced by a ray passing through a cluster calculated only with $4D$ general relativity (GR). The presence of Platonic KK modes acts exactly as a dark halo of cold dark matter.

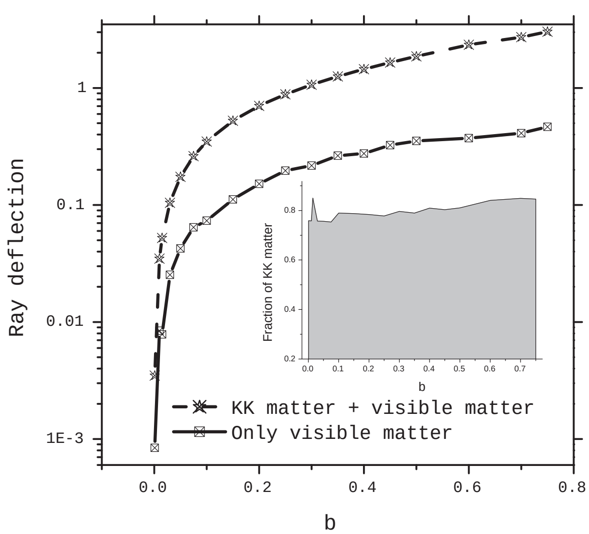


FIG. 10: Numerical integration of the gravitational lensing deflection produced by a cluster living in a universe endowed with UED. In our model, the KK matter constitutes $(79.5 \pm 3.3)\%$ of the total matter in a spherically symmetric galactic cluster.

verified by lensing observations [6] that cluster of galaxies are composed of three main components: $\sim 5\%$ in mass is the optically luminous baryonic matter in hundred of bright galaxies; $\sim$ 10–15% is in the form of a bright X-ray inter-cluster gas; and the remaining $\sim$ 80–85% is some sort of non-baryonic "missing mass". The explanation of our results is that such non-baryonic "missing mass" can be explained by the presence of KK modes distributed in the cluster, appearing in accordance to the Platonic Model [34].

IX. CONCLUDING REMARKS

At the present paper, the possibility to construct a galactic disk embedded in a multidimensional space-time is investigated. Here, following the observational evidence of DM in galaxies and clusters, we assume that the DM particle comes from extradimensional modes, or (KK modes), although initially we do not know if those modes are distributed isotropically on the galaxy disk or are described by a spheroidal halo (as preached by CDM cosmologies). Assuming first the "isotropic on the disk" scenario, we investigated solutions of Einstein field equations in axially symmetric configurations in D dimensions to construct a disk model composed of Kaluza-Klein particles. After demonstrate that vacuum Einstein equations can be written as a set of Laplace's equations for any number D of dimensions, we showed that if D is even, then the sum of extradimensional pressures cancel and the total pressure depends only of $4D$ components. Thus, the density profiles are practically the same as in the $4D$ case and we find that the extra dimensions do not affect the density and the azimuthal and radial pressures. As the simplest example, in a six dimensional space-time a disk was constructed by solving the vacuum Einstein equations for an extension of the Weyl's metric. In particular, was studied a disk constructed from Schwarzschild and Chazy-Curzon solutions with a simple extension for the extra dimensions. Two integral constants of motion from projection of extradimensional particle velocities are the free parameters of the model. We had prevented the *ad hoc* adjustment of such parameters with observed rotation curves, preferring to investigate values where the disk becomes stable. The stability is achieved when the disk is Newtonian (where such parameters are null) or for a tiny range of values that astonishingly makes the circular geodesics fit with great precision the rotation curves of many spiral galaxies. The stability calculation is done using both the Rayleigh criterion and a perturbative approach. We compare such results to well succeeded astrophysical dark matter profiles and demonstrate that our predictions give the same gravitational lensing as does a dynamically successful dark halo model.

A. On NFW profiles

Current chiefly astrophysical models used to achieve cosmological results for the evolution of cluster of galaxies are the cold dark matter (CDM) approaches [7], where the "missing matter" problem is solved by using a dark halo of exotic postulated particles surrounding galaxies. The Navarro-Frenk-White profile [56] is usually the approach employed. *A priori* the construction of the halo needs no physical explanations and the profiles derived phenomenologically are only useful to simulate but not to explain. In general it is delegated to particle physicists the effort to furnish the explanations. And actually many particle physicists are satisfied with the manner how the CDM cosmologies work: artificial simulations of postulated particles included by hand to explain the large scale structure. The success of cosmological simulations, mainly obtained by the ΛCDM models, deserves important astrophysically phenomenological honor mentions, but as pointed, the artificial scientific fashion of the model to explain large scale physics is outrageous deficient. NFW construct a DM halo with some peculiar density profile. The paradigm is that this halo had been formed at the beginning of structure formation and the non-baryonic candidates for particles in such models range e.g. from primordial black holes [9] to elementary particles relics left over from the early universe (e.g. WIMPS [10]). There are potential problems for the ΛCDM model, as for most other versions of CDM models. On large scales (clusters, superclusters, cosmological filaments), CDM simulations have proven very successful, while on smaller (galaxy) scales, they were faced with a number of problems. A first, the cusp problem, the density profiles of simulated galaxies possess a central cusp, according to the NFW law, while this is not observed in many galaxies, in particular in those with low masses (so-called dwarf galaxies). A second problem is that simulated galaxies come out too small, or have too little angular momentum. Another problem is the satellite one: simulated DM halos possess a host of substructure. However, one observes but few satellite galaxies around larger ones (e.g our Milky Way).

Here our model does not assumes an *ad hoc* halo, i.e. a different paradigm as that preached by CDM cosmologies. The concentration of matter forming structures arises naturally from the superposition of matter density and KK modes. The propose is not to fit exactly the astrophysical data, but to offer a dynamical explanation to anomalies in rotation curves and gravitational lensing regime. Simple models of a disk and a cluster are constructed and we show that: the observational density profiles of our galaxy are the same as observed for a $4D$ matter distribution; the stable rotation curves fit the plateau region of many spirals; the lensing regime gives the same gravitational lensing as does a dynamically successful dark halo model. Here we had focus on dynamics, but astrophysical models could be constructed using the enrichment of our results by the inclusion of gas/plasma phenomenology, asymmetries on the disk plane, central massive black holes, chemodynamics of a bulge and the role of stars and bars.

On cosmological issues it is well known that DM appears in the primordial nucleosynthesis calculation as the same fraction presented by measurements of cosmological microwave background. The acoustic peaks plus the concordance observations of supernovae and clusters show that this fraction is $\Omega_{DM} \sim 0.25$, with $\Omega_b \sim 0.5$ being due to baryons and $\Omega_\Lambda \sim 0.7$ due to dark energy. Here the cosmological problem is left open, although important topics about e.g. braneworld cosmology are well developed in a long range of works. The existence of two or more extra dimensions certainly has the potential to signicantly influence the $4D$ evolution of the universe, thus altering standard cosmology. To see how essentially occurs such modifications, follow for instance the references in [58, 59].

B. Discussions on LHC direction

The observation of rotation curves of galaxies and gravitational lensing in galactic clusters, as explained, is directly related to the dark matter problem. According to our results, maybe "dark matter" could be interpreted by a conception of projected extra dimensional particles/fields on the 4D disk space-time. Eqs. (5.107) and (5.108) show that the projection of extra dimensional rates is the determinant factor to obtain a deviation from a Newtonian profile. It is a kind of Kaluza-Klein modes imprint in astrophysical scales. The main point is that the model confirms the existence of dark matter, but it is not distributed as a halo. Although the main purpose of the present work is to construct a top-down model and not to shed light on a fundamental theory we can illustrate considering the possibility that our model could constrain a Kaluza-Klein dark matter particle to be tested at Large Hadron Collider (LHC) in next years. Consider for simplicity that Kaluza-Klein dark matter particle in Universal Extra Dimensions could be considered as a fundamental particle, where is included also the possibility that a such particle could decay in others. Following [2], one can derive a Lagrangian from the compactification of the extra dimensions and connect such compactification to the cross section of the particle. Using the generic notation x^α, $\alpha = 0, 1, ..., 3 + \delta$ for the coordinates of the $(4 + \delta)$-dimensional space-time (different from the usual non-compact space-time coordinates, x^μ, $\mu = 0, 1, 2, 3$), and the coordinates of the extra dimensions, y^a, $a = 1, ..., \delta$, the 4-dimensional Lagrangian can be obtained by dimensional reduction from the $(4 + \delta)$-dimensional theory,

$$\mathcal{L}(x^\mu) = \int d^\delta y \Bigg\{ -\sum_{i=1}^{3} \frac{1}{2\hat{g}_i^2} \mathrm{Tr}\left[F_i^{\alpha\beta}(x^\mu, y^a) F_{i\,\alpha\beta}(x^\mu, y^a)\right] + \mathcal{L}_{\mathrm{Higgs}}(x^\mu, y^a)$$
$$+ i\left(\overline{\mathcal{Q}}, \overline{\mathcal{U}}, \overline{\mathcal{D}}\right)(x^\mu, y^a)\left(\Gamma^\mu D_\mu + \Gamma^{3+a} D_{3+a}\right)(\mathcal{Q}, \mathcal{U}, \mathcal{D})^\top (x^\mu, y^a)$$
$$+ \Big[\overline{\mathcal{Q}}(x^\mu, y^a)\left(\hat{\lambda}_\mathcal{U}\mathcal{U}(x^\mu, y^a) i\sigma_2 H^*(x^\mu, y^a) + \hat{\lambda}_\mathcal{D}\mathcal{D}(x^\mu, y^a) H(x^\mu, y^a)\right) + \mathrm{h.c.}\Big]\Bigg\} . \quad (9.133)$$

Here $F_i^{\alpha\beta}$ are the $(4+\delta)$-dimensional gauge field strengths associated with the $SU(3)_C \times SU(2)_W \times U(1)_Y$ group, while $D_\mu = \partial/\partial x^\mu - \mathcal{A}_\mu$ and $D_{3+a} = \partial/\partial y^a - \mathcal{A}_{3+a}$ are the covariant derivatives, with $\mathcal{A}_\alpha = -i\sum_{i=1}^3 \hat{g}_i \mathcal{A}^r_{\alpha i} T^r_i$ being the $(4+\delta)$-dimensional gauge fields. The piece $\mathcal{L}_{\mathrm{Higgs}}$ of the $(4+\delta)$-dimensional Lagrangian contains the kinetic term for the $(4+\delta)$-dimensional Higgs doublet H, and the Higgs potential. The $(4+\delta)$-dimensional gauge couplings $\hat{g}_i$, and the Yukawa couplings collected in the 3×3 matrices $\hat{\lambda}_{\mathcal{U},\mathcal{D}}$, have dimension $(\mathrm{mass})^{-\delta/2}$. The fields $\mathcal{Q}, \mathcal{U}$ and $\mathcal{D}$ describe the $(4+\delta)$-dimensional fermions whose zero-modes are given by the 4-dimensional standard model quarks. A summation over a generational index is implicit in Eq. (9.133). For example, the 4-dimensional, third generation quarks may be written as $\mathcal{Q}_3^{(0)} \equiv (t,b)_L$, $\mathcal{U}_3^{(0)} \equiv t_R$, $\mathcal{D}_3^{(0)} \equiv b_R$. The kinetic and Yukawa terms for the weak-doublet and -singlet leptons, $\mathcal{L}$ and $\mathcal{E}$, are not shown for brevity. The gamma matrices in $(4+\delta)$ dimensions, Γ^α, are anti-commuting $2^{k+2} \times 2^{k+2}$ matrices, where k is an integer such that $\delta = 2k$ or $\delta = 2k+1$. Chiral fermions exist only when δ is even, and correspond to the eigenvalues ±1 of $\Gamma^{4+\delta}$. The space-time is described as $M^4 \times T^\delta$, where the extra dimensions can be compactified in a T^δ tori. The $6D$ example of Section IV has a tori of radii $r_x = e^{\nu/2}$ and $r_y = e^{-\nu/2}$. Asymptotically the compactification tori radius is constant and it is obtained by $r_c = \sqrt{r_x^2 + r_y^2}$, and the periodic dimensions $x^k \cong x^k + 2\pi r_c$, $6-\delta \le k \le 5$. The compactification is obtained by imposing the identification of two pairs of adjacent sides of the torus retangle. In a pure UED model the compactication scale arises simply from the need to produce the right amount of DM. Here, a modified UED model, the compactification scale arises from the dynamics of galaxies and from clusters lensing. A detailed investigation about the compactification of the model will be done in future works, although our preliminary results indeed strength the possibility that our model could constrain a Kaluza-Klein dark matter particle to be tested at Large Hadron Collider (LHC), where the proposed energy range is 1–14 TeV, and the main signals of KK DM will be several $t\bar{t}$ resonances.

Acknowledgments

C.H. C.-A. thanks IUPAP and CAPES and P.S.L thanks CNPq and FAPESP for partial financial support.

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